Algorithmic Trading

Optimal Execution Models and the Real World

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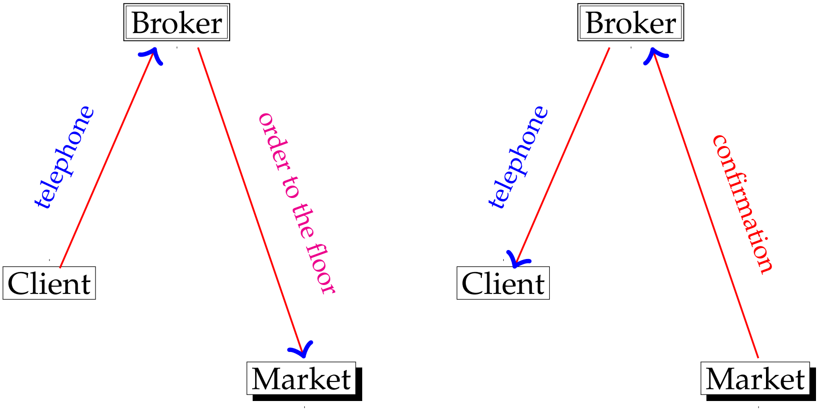
Table of Contents

In the Past: High-Touch Brokers



Algorithmic Trading (AT)

Definition

- The use of programs (algorithms) and computers (automation) to generate and submit orders in electronic markets.

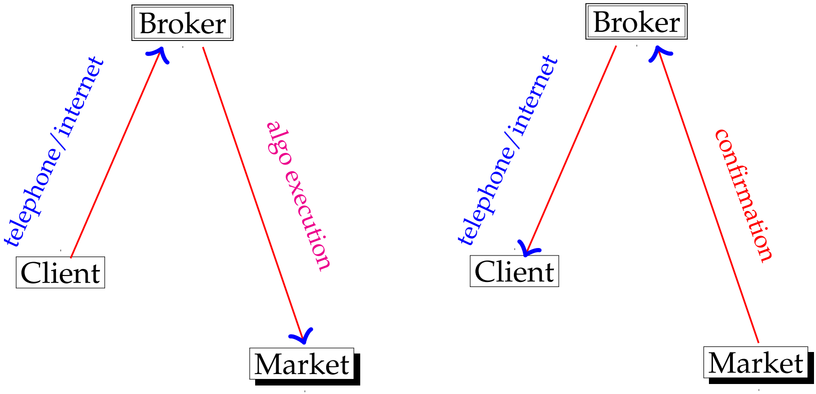
- Origin: U.S. institutional investors in late 1990s needed the tools to deal with major changes:
  - electronic markets: electronic communications networks
  - alternative trading systems, dark pools
  - decimalization (reduction) of tick size
  - reduction of commissions and exchange fees

- Today, brokers compete actively for the commission pool associated with algorithmic trading around the globe.

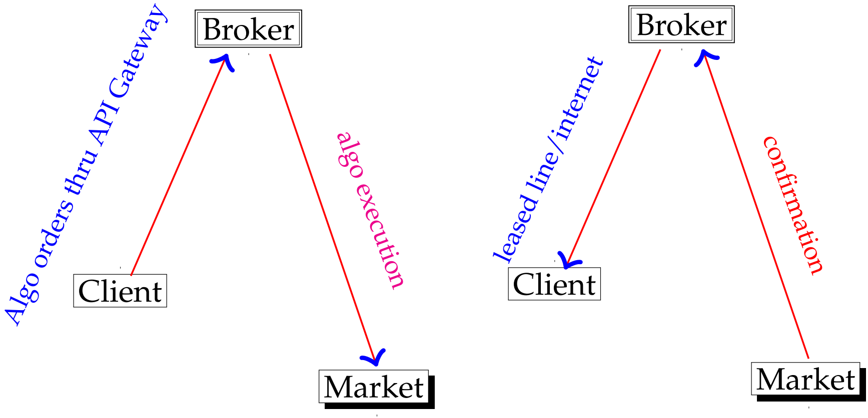
Characteristics of Algorithmic Trading (AT)

- Non-discretionary
  - Purely procedural or mechanical
  - Emotionless
- Algorithmic
  - Trading rules
  - Artificial intelligence, machine learning, cyborg
- Data driven
  - Tick-by-tick trades and quotes, news, fundamentals etc.
  - Live, real-time feeds absolutely necessary
- Automatic
  - Automation: major investment in high-tech IT infrastructure (ultra high-frequency, low latency, direct market access, co-location)
  - If the algorithmic strategy is not speed-sensitive, automation is not absolutely necessary

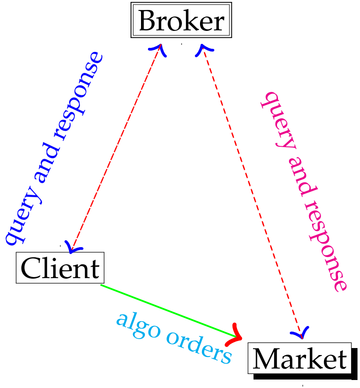
Low-Touch Electronic Market



Electronic Market with API Gateway



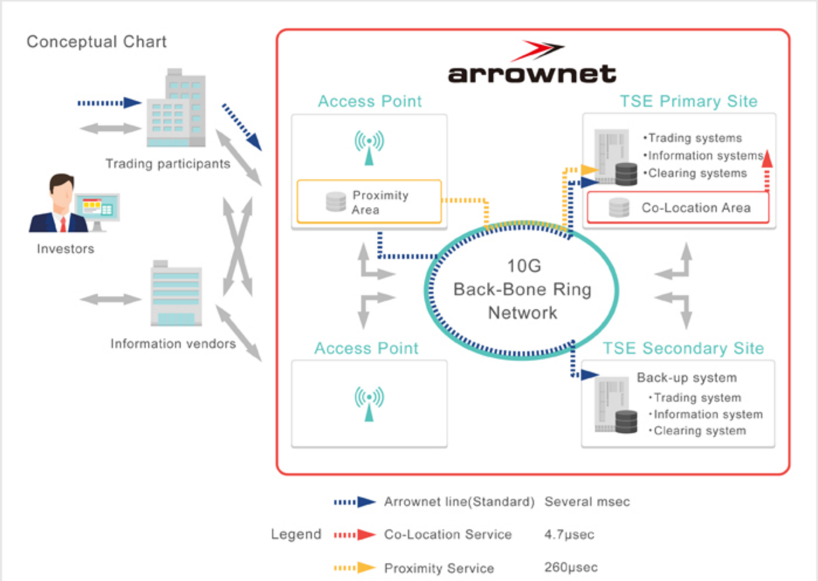
Direct Market Access (DMA) and Co-loc



- No touch
- Co-location: the trading algo sits in a server next to and connected to the exchange's matching engine
- Low latency

Arrownet

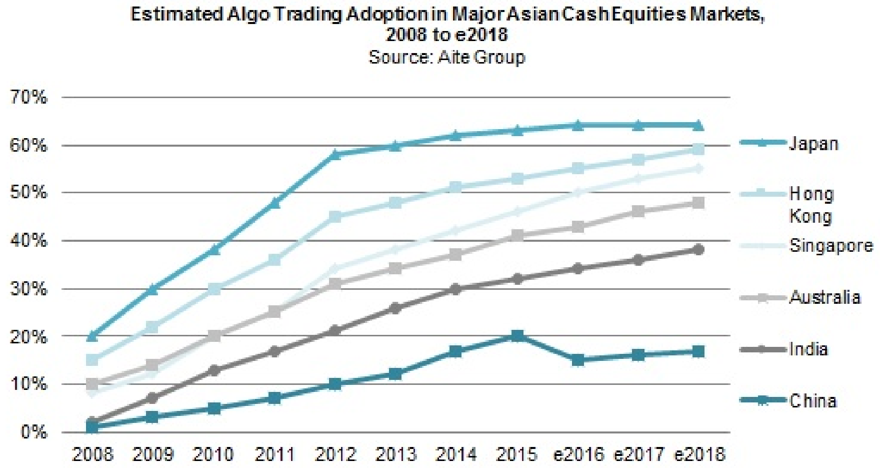
Source: JPX



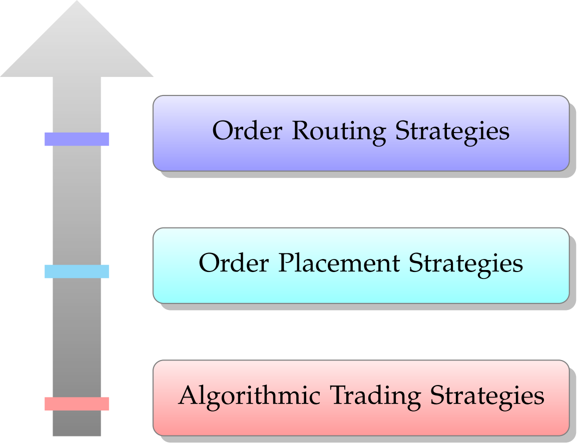
Market Fragmentation: Case Study of Japan

- Exchanges
  - JPX Group
    - First Section
    - Second Section
    - Jasdaq
    - Mothers
  - Nagoya Stock Exchange
  - Sapporo Securities Exchange
  - Fukuoka Stock Exchange
- Proprietary trading system (PTS)
  - Chi-X Japan
  - SBI Japannext
- Exchanges versus PTS: More then one venue to transact the same stock.

Adoption of AT in Asia



Three Building Blocks of AT



Two Major Categories of AT Strategies

Alpha Generation, Buy Side

- Location arbitrage
- Triangular arbitrage
- Statistical arbitrage
  - Momentum
  - Mean reversion
- Market making

Two Major Categories of AT Strategies (cont'd)

Brokerage Execution, Sell Side

- Services rendered by sell side to and for the buy side
- Time-weighted average price (TWAP)
- Volume-weighted average price (VWAP)
- Percentage of volume (POV)
- The Almgren-Chriss algorithm
- Liquidity-seeking

Institutional Brokerage Execution

- Since the 2008 financial crisis, “alpha” generation is strongly curtailed for major investment banks.
- Nevertheless, institutional clients need to trade large amounts of stocks, larger than what the market can absorb without impacting the price.
- To prevent slippage, a large (parent) order must be sliced into many smaller child orders.
- What is the algorithm to bring about an optimal solution?
- Providing trade execution services for institutional clients is a BIG business these days.

First-Generation AT for Brokerage Execution

- **TWAP**: Equal amount of shares or contracts in each time interval.
- But the trading volume exhibits a U-shape pattern from opening to closing.
- **VWAP**: Divide the trading session into 5- to 15-minute interval. Predict intra-day trading volume and price range for each interval by some analytics. In each time-interval, execute an amount proportional to the predicted volume for that interval.
- **POV**: Target a defined fraction of the actual volume for each time interval. The intention is to keep the trading activity in line with total volume. Trade at a constant percentage participation rate.

Volume Weighted Average Price

- Let $p_i, v_i, i = 1, 2, \ldots n$ be the prices and volumes, respectively for which n child orders are filled.
- Total volume traded is
$$v = \sum_{i=1}^{n} v_i.$$
- With $w_i = v_i/v$ being the weight, The VWAP price is, by definition,
$$P_{\text{vwap}} := \sum_{i=1}^{n} w_i p_i = \sum_{i=1}^{n} \frac{v_i}{v} p_i = \frac{\sum_{i=1}^{n} p_i v_i}{v} = \frac{\text{Dollar Volume}}{\text{Volume}}$$
- VWAP is the break-even price.

VWAP Algo Trading Setup

- A trader must buy $v(T)$ shares by trying to get the average purchase price as close as possible to the market VWAP during the trading period from time 0 to time T.
- At time 0, $v(0) = 0$ and at time T, $v(T)$ is the number of shares to buy.
- The trader's own VWAP is expressed as

$$\text{vwap} = \frac{\int_0^T P(s)\, dv(s)}{v(T)}.$$

- With $M(t)$ denoting the volume done by other traders, the market VWAP is

$$\text{VWAP} = \frac{\int_0^T P(s)\, d[M(s) + v(s)]}{M(T) + v(T)}.$$

Objective Function and Approximation

- Objective of the algorithm is to minimize the expected error (see [Konishi, 2002]):

$$\min_{v(t)} \mathbb{E}\left(\left(\text{VWAP} - \text{vwap}\right)^2\right)$$

- Definition: Percentage of remaining volume

$$X(t) := \frac{M(T) - M(t)}{M(T)}; \qquad x(t) := \frac{v(T) - v(t)}{v(T)}$$

- Change of objective to

$$\min_{x(t)} \mathbb{E}\left(\left(\int_0^T \left(X(t) - x(t)\right) dP(t)\right)^2\right)$$

Assumption

- The stock price process is assumed to be a martingale

$$dP(t) = \sigma\, dB(t)$$

- Given that σ^2 is a constant, the objective becomes

$$\min_{v(t)} \int_0^T \mathbb{E}\left(\left(X(t) - x(t)\right)^2\right) dt.$$

- Let t_k, $k = 1, 2, \ldots, v(T)$ denote $v(T)$ discrete execution times, and define $t_0 = 0$ and $t_{v(T)+1} = T$. Then $x(t)$ is a step function with $v(T) + 1$ values:

$$x(t) = 1 - \frac{k}{v(T)} \qquad \text{if } t_k < t \leq t_{k+1}, k = 0, 1, \ldots, v(T).$$

- Minimization becomes the problem of approximating a function, continuous almost everywhere, with a step function $x(t)$.

Solution

- [Konishi, 2002] shows that an optimal schedule is

$$x^*(t) = \mathbb{E}\big(X(t)\big) \pm \frac{1}{2v(T)}.$$

- The optimal schedule is to overshoot $\mathbb{E}\big(X(t)\big)$ by $1/(2v(T))$, which is a small amount if the number of shares to trade is large.
- It is crucial to forecast the mean $X(t)$ at each time period t, which is the proxy for $\mathbb{E}\big(X(t)\big)$.
- Therefore, you need to forecast the total volume $M(T)$ of the market, *and* the dynamics of the market volume $M(t)$!

Second-Generation AT for Brokerage Execution

- [Almgren and Chriss, 2000]'s mathematical modeling
- Consider liquidating X shares. Let the number of shares yet to be liquidated be $x_t, t = 0, 1, \ldots, T$. So

$$x_0 = X, \qquad x_T = 0,$$

and

$$x_s \leq x_t \qquad \text{if } s > t.$$

- Accordingly, $-\left(x_{t+\epsilon} - x_t\right)$ is the number of stocks sold in the time interval $(t + \epsilon, t)$. Hence

$$x_{t+\epsilon} - x_t \longrightarrow dx_t, \qquad \text{as } \epsilon \to 0.$$

An Observation

Proceeds from Liquidation

- The stock price S_t is assumed to be a martingale (i.e., no drift).
- Show that the expected proceeds from liquidation are

$$-\mathbb{E}\left(\int_0^T S_t\, dx_t\right) = S_0 X.$$

Proof

- Integration by parts

$$\int_0^T S_t\, dx_t = S_T x_T - S_0 x_0 - \int_0^T x_t\, dS_t$$

- Since S_t is a martingale, we have $\mathbb{E}\left(\int_0^T x_t\, dS_t\right) = 0$.

Intuition

- The stock price S_t is a martingale, implying a memory-less random walk, i.e.
$$\mathbb{E}\big(S_t\big) = S_0$$
for all $0 < t \leq T$.

The average value of a random walk's position is the starting point!

- The amount X to sell is exogenous to S_t.
- At time 0, the fund manager decides to sell. The market price of the stock is S_0. It follows that the market value of the stock holding in dollars is none other than S_0X at time 0.

Implementation Shortfall

- With $x_T = 0$ and $x_0 = X$, the integration by parts results in

$$\begin{aligned}\int_0^T S_t\,dx_t &= -S_0 X - \int_0^T x_t\,dS_t \\ &= \mathbb{E}\left(\int_0^T S_t\,dx_t\right) - \int_0^T x_t\,dS_t.\end{aligned}$$

- Hence

$$\int_0^T S_t\,dx_t - \mathbb{E}\left(\int_0^T S_t\,dx_t\right) = -\int_0^T x_t\,dS_t.$$

- What is the intuitive interpretation of the above equation?

  Answer: ______________________________

Almgren-Chriss' Assumptions

- The price S_t is a drift-less (arithmetic) Brownian motion B_t:

$$dS_t = \sigma\, dB_t$$

  with constant (intra-day) volatility σ.

- The price at which a sale takes place is not S_t but P_t given by

$$P_t = S_t + \eta\, v_t,$$

  where the rate of trading is

$$v_t \equiv \dot{x}_t := \frac{dx_t}{dt},$$

  i.e., x_t is a (deterministically) differentiable function of t.

Tutorial Questions

Q1: Expected cost of Transaction

Since $dx_t = v_t\,dt$, consider

$$\mathcal{K} := \mathbb{E}\left(\int_0^T P_t\,dx_t\right) = \mathbb{E}\left(\int_0^T P_t\,v_t\,dt\right)$$

Show that

$$\mathcal{K} = \eta \int_0^T v_t^2\,dt - S_0 X$$

under Almgren-Chriss' assumptions.

Tutorial Questions (cont'd)

Q2: First-Order Condition

Show that minimization of $\mathcal{K}$ with respect to x_t results in the first-order condition:

$$\frac{dv_t}{dt} = \frac{d^2 x_t}{dt^2} = 0.$$

Q3: Optimal Trading Schedule

Solve the first-order condition in Q2 with the "boundary conditions" $x_0 = X$ and $x_T = 0$ and show that the optimal liquidation strategy is

$$x_t^*(t) = X\left(1 - \frac{t}{T}\right),$$

for $t = 0, 1, \ldots, T$.

Tutorial Questions (cont'd)

Q4: Quiz

Which of the three strategies does x_t^* correspond to?

1. TWAP
2. VWAP
3. POV
4. None of the above

Trading Risk

- Trading risk is captured by

$$\mathcal{V} := \mathbb{V}\left(\int_0^T x_t\, dS_t\right) = \sigma^2 \int_0^T x_t^2\, dt.$$

- Main idea of [Almgren and Chriss, 2000] is to minimize the mean *and* variance with λ being the Lagrange multiplier:

$$\begin{aligned}\mathcal{A} &:= \mathbb{E}\left(\int_0^T P_t v_t\, dt\right) + \lambda \mathbb{V}\left(\int_0^T x_t\, dS_t\right)\\ &= \mathcal{K} + \lambda\sigma^2 \int_0^T x_t^2\, dt\\ &= \int_0^T \left[\eta\left(\frac{dx_t}{dt}\right)^2 + \lambda\sigma^2 x_t^2\right] dt\end{aligned}$$

Physics Envy?

- The first term corresponds to the kinetic energy:

$$\eta = \frac{1}{2}m; \qquad v_t = \frac{dx_t}{dt} = \dot{x}_t,$$

  giving rise to $\frac{1}{2}mv_t^2$.

- The second term $\lambda\sigma^2 x_t^2$ corresponds to the (negative) potential energy.
- $\mathcal{A} = \mathcal{K} - (-\mathcal{V})$ is the action of a mechanical system!
- It follows that $\mathcal{A} =: \int_0^T \mathcal{L}\, dt$, where $\mathcal{L}$ is the Lagrangian.
- Mean-variance optimization is equivalent to solving the Euler-Lagrange equation of least action!

Analytical Solution

- The Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_t} - \frac{\partial \mathcal{L}}{\partial x_t} = 0.$$

- For the [Almgren and Chriss, 2000] Lagrangian, i.e., $\mathcal{L} = \eta \dot{x}_t^2 - \big(- \lambda \sigma^2 x_t^2 \big)$, it is

$$2\eta \ddot{x}_t - 2\lambda\sigma^2 x_t = 0,$$

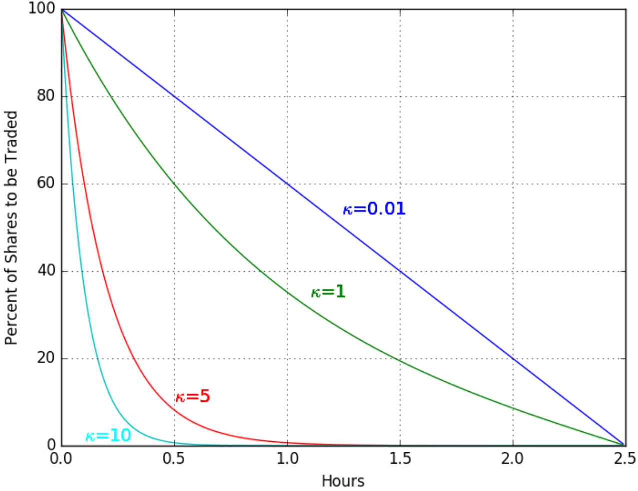
i.e., $\ddot{x}_t = \kappa^2 x_t$, where

$$\kappa^2 = \frac{\lambda\sigma^2}{\eta}$$

- The solution that satisfies the boundary conditions is

$$x_t^\star = X\frac{\sinh\big(\kappa(T-t)\big)}{\sinh(\kappa T)}$$

Optimal Schedule for Different Urgency κ



Tutorial

- What is the strategy corresponding to $\kappa = 0$?

  Answer: ________________________________

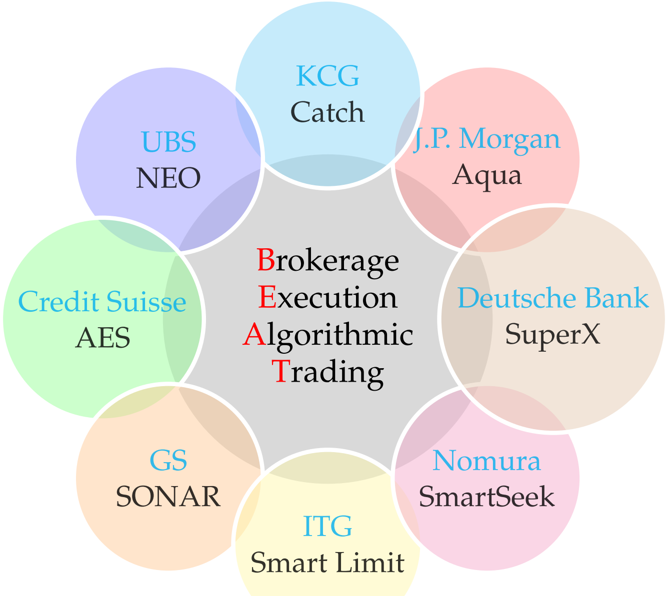
- Derive the rate of trading v_t^* for the optimal $x_t^\star$.

- If the volatility σ is large, all else being equal, do you want to trade faster?

Third-Generation: Liquidity-Seeking AT

- Multi-period optimization of trade scheduling
- Access to all available liquidity, both dark and lit
- Inclusion of order placement and order routing algorithms
- Urgency specification
    - Low: dark, passive
    - Medium: combination of dark and opportunistic participation
    - High urgency: high participation rate
- Optimization of probability of fill (execution certainty), invisibility (stealth), price improvement (passiveness) trading cost (implementation shortfall), and price certainty (risk) subject to client's specification of urgency.

BEAT Systems and Business



Lit and Unlit Orders

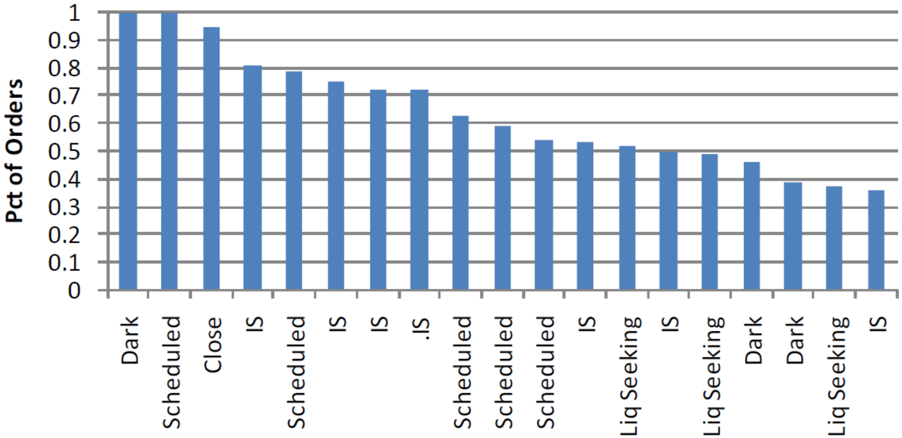
- In the past, “upstairs” markets and crossing networks allowed institutional block trades to execute.
- Dark pools are the results of technological advancement, intense competition, and new regulatory requirements.
- Most brokers support iceberg orders for lit exchanges, resulting in dark liquidity.
- Is there a smart way to ping for dark liquidity?

ITG's Classification of Algo Trading Strategies

- Close: Trading at the close
- Dark: Dark pool and unlit orders
- Scheduled: TWAP, VWAP, POV (Participation)
- Implementation Shortfall (IS): Inspired by and generalization of [Almgren and Chriss, 2000]
- Liquidity-Seeking

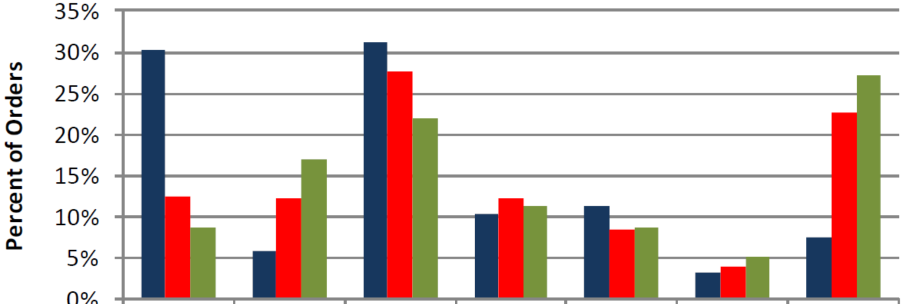
Preferred BEAT of 20 Buy-Side Desks

[Domowitz and Yegerman, 2011] Sample Period: Q4 2009 to Q4 2010 Q4



Order Placement Strategies

[Domowitz and Yegerman, 2011] Sample Period: Q4 2009 to Q4 2010 Q4



	CLOSE	DARK	IS	LIQ SEEKING	PARTICI PATION	TWAP	VWAP
LT.5%	30%	6%	31%	10%	11%	3%	8%
.5%-1%	13%	12%	28%	12%	9%	4%	23%
1%-5%	9%	17%	22%	11%	9%	5%	27%

Order Placement Strategies (cont'd)

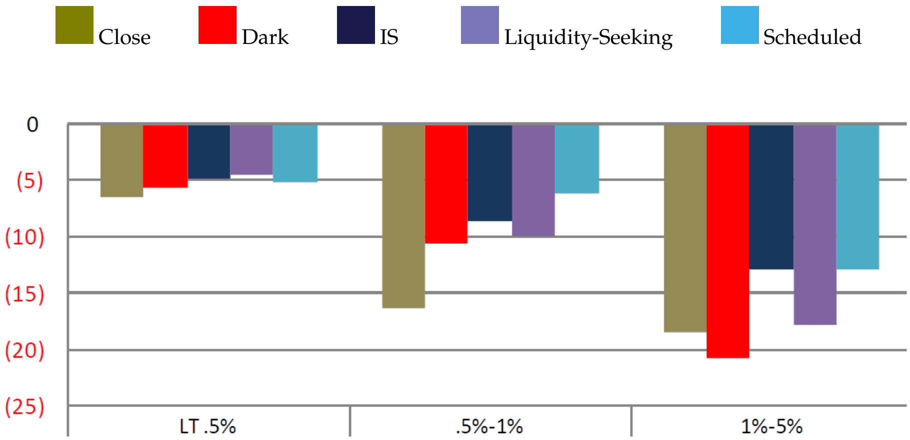
- Demand for liquidity: percentage of median daily volume (MDV) for the parent order
  1. Less than (LT) 0.5% of MDV
  2. 0.5% to 1% of MDV
  3. 1% to 5% (and rarely above) of MDV

- During the "quiet" period (Q4 2009 to Q4 2010 Q4), Almgren-Chriss' Implementation Shortfall (IS) strategy was most popular.
  - 81% of orders were executed by IS

- VWAP is still dominantly used (58% of orders) by some trading desks.

Trading Strategy Costs By Demand For Liquidity

[Domowitz and Yegerman, 2011] Sample Period: Q4 2009 to Q4 2010 Q4



Takeaways

- BEAT is a big business and R&D of algorithmic trading systems will continue.
- Generalization to account for stochastic liquidity and volatility (see [Almgren, 2012])
- Permeation of BEAT systems to FX, futures and options, and fixed income securities
- Asia-Pacific markets still have a lot of room to grow the BEAT business.

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