ALGORITHMIC FINANCE

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CONTENTS

CHAPTER 1

RETURNS

We start from the very beginning of quantitative analysis of investments, which is none other than the data.

1.1 Time Series

Suppose you are interested in a financial asset such as a publicly listed stock of a company. In the digital age, it has become much easier to gain assess to news and reports about the company. You examine the company's past and current financial strength, corporate governance, future potential, and so on. You also look at what other analysts and investors are saying about the company. Obviously, you look at the stock price, which reflects, at any given time, the level of demand for the shares issued by the company.

The initial public offering of a company ushers in birth of stock prices. By market convention, the last traded price of a business day is regarded as the stock price. Supposedly, it reflects the market evaluation of the company's worth per share. The fact that stock price moves in a seemingly random fashion suggests that the market, comprising investors and speculators, are not very sure about the company valuation. Literally, the share price can change from one transaction to another. what it means is that you can sample stock price as and when it occurs, or at a specified time such as the closing time of the exchange. Whichever, if you sample consistently, you obtain a chronologically arranged sequence of prices.

To make things precise, we introduce the notion of event, which is a fundamental concept in probability theory. The events of interest in the financial market are many. It ranges from a company announcement, release of monetary policy, announcement of macroeconomic indicator, down to the transaction of a stock. You can record the last traded price of a stock according to the clock time, say 4 PM local time every working. You can also record the transaction as and when it occurs. In this case, the time at which the transaction occurs is said to be business time.

Definition 1.1 **Regular sampling** is a data collection scheme that is based on the clock time. **Irregular sampling**, on the other hand, is based on the businss time, which is the arrival time of an event that gives rise to a set of numbers to be recorded as a sample. ■

Throughout the book, we use the symbol t to denote the time. For regular sampling, t is the clock time *by* which the last transaction of the trading day or trading session takes place. For irregular sampling, t is the time *at* which a trade occurs. Each trade is identified by a serial number t, which indicates its chronological order in the time series. Though we refer to financial transaction for specificity, in general, the subject of interest can be any event such as weather forecast announcement.

Regardless of whether the observations are collected by clock time or business time t, we have a formal definition of a sequence of quantities.

Definition 1.2 **Time series** is defined as a chronologically arranged sequence of quantities sampled by a sampling scheme consistently. The time series is denoted as P_t, for $t = 1, 2, 3, \ldots, T$, where T is the last observed value in the sample. ■

It is important to emphasize that the same sampling scheme is applied consistently; you stick to one sampling scheme throughout when recording the time series.

Notice that we have implicitly assume that the time series is **discrete** with respect to time t. For empirical analysis using numerical algorithms, the time series must be discrete. From the modeling standpoint, however, it is often convenient to consider **continuous** time series, which is a mathematical construct in the limit when the time interval is infinitesimally small.

EXAMPLE 1.1

We list a few examples of time series in Table 1.1. We have listed two major events for a publicly listed company as examples. Earnings announcement is a highly watched event for analysts and investors. In the US, companies are obliged under regulations to announce their financial report. One of the most important numbers

Table 1.1 Examples of Time Series.

Event	Quantity	Time	Remarks
Transaction	Price	Clock	Usually last traded price, daily
Transaction	Intraday Price	Clock	Usually 5-minute interval
Transaction	Tick-by-tick Price	Business	Highest frequency
Earnings Announcement	Earnings per share	Business	Scheduled
Company Distribution	Dividend per share	Business	Ex date
US Employment Situation	Nonfarm payrolls	Clock	Every first Friday of the month
US ISM Manufacturing Index	Index level	Clock	First business day of the month.

is earnings per share (eps). It shows how much a company has earned over a period of time. The other event is what is known as distribution of earnings to the shareholders. Usually, the distribution is in the form of cash. At times, it can be in the form of shares, and other alternatives. A key date for this event is known as the ex date. If you are an investor and your name is in the company's registry of shareholders before the ex date, then you are entitled to receive the distribution.

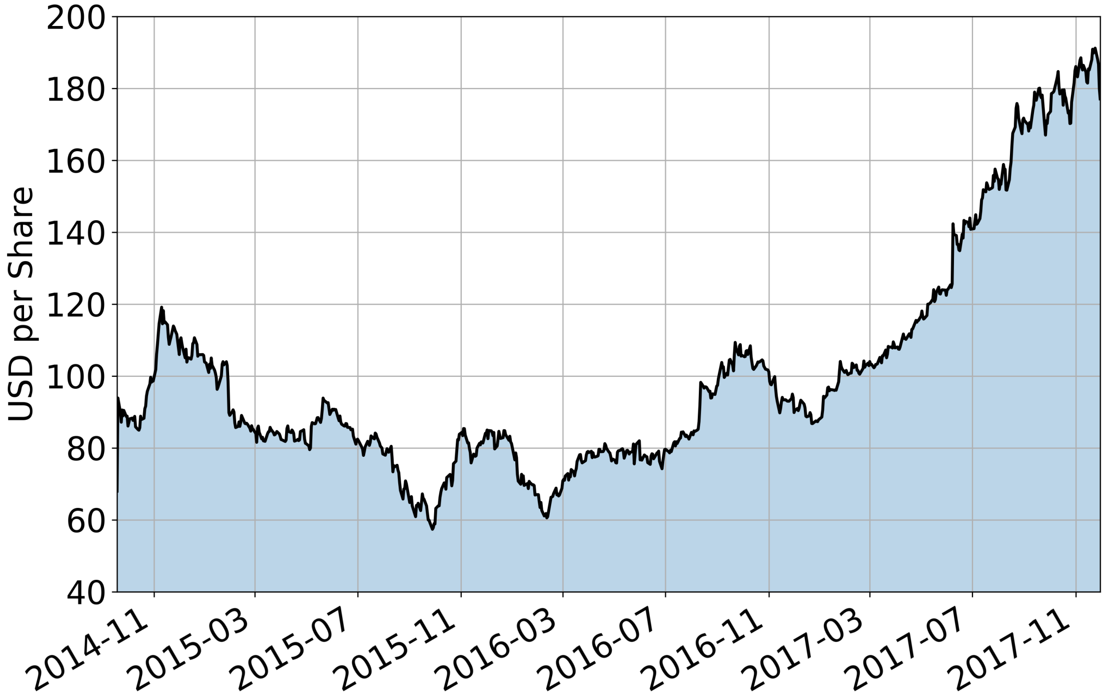
Listed also in Table 1.1 are two of the most highly watched macroeconomic news. The US Employment Situation shows the number of nonfarm jobs created. Its significance is underscored by the fact that it is a monthly indicator of aggregate economic activity, as it encompasses all major sectors of the economy. The The ISM manufacturing composite index indicates overall trend of manufacturing activities. It provides insight on commodity prices and clues regarding inflation. ■

By definition, Time series is a discrete sequence of chronologically ordered numbers. It comes with no surprise that everyone has difficulty looking at just numbers. Thus, it is necessary to present these ordered numbers in a visually intuitive and insightful fashion. **Data visualization** is a sub-branch of **data science**. A key application of data visualization is to bring out different aspects embedded or hidden in the data. The simplest data visualization method is to plot the time series.

EXAMPLE 1.2

As a holding company based in Beijing, China, Alibaba provides internet infrastructure, e-commerce, online financial, and internet content services through its subsidiaries worldwide. Its initial public offering (IPO) is the largest ever in the history of stock market. Traded by the ticker symbol of BABA, its daily time series of stock prices is plotted in Figure 1.1. Often, Figure 1.1 is referred to as **line chart**

Figure 1.1 Stock Price of Alibaba Group Holding Ltd Since IPO.



for a single time series.

The IPO subscription price is \$68.00 per share. On September 19, 2014—first day of trading of Alibaba shares—the last traded price is \$93.89, which is substantially higher. After about a month of heading lower, it starts to rise about \$100 per share and eventually reaches \$120. From November 2014, the stock price is on the downward trend, and eventually dips below \$60. But from January 2017, the stock is finally on the trajectory of bull run. reaching close to \$180 on end of November 2017. Surely, the line chart is more palatable than just looking at the ordered sequence of prices: \$68.00, \$93.89, \$89.89, \$87.17, ..., \$186.69, \$179.91, \$177.08. ■

1.2 Multiple Time Series

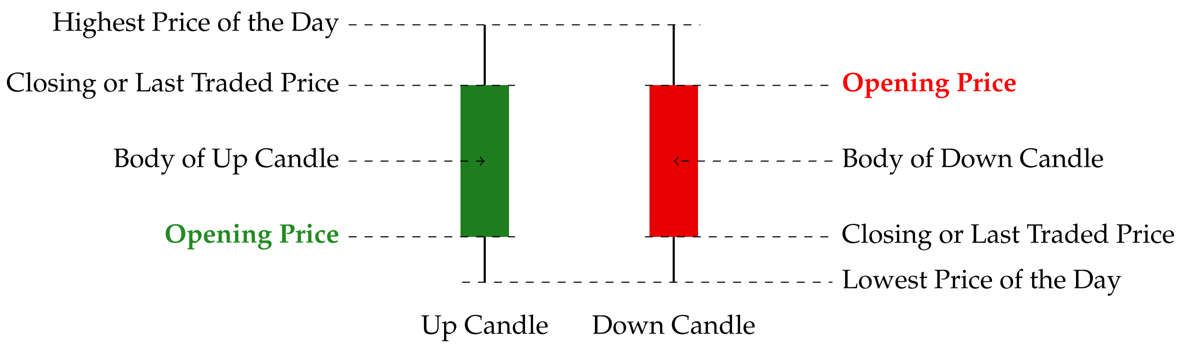
At times, not only can you get the last traded price, but also the opening price, the highest price of the day, and the lowest price of the day. These three other prices allow you to see at least the following features of trading about an asset of interest for any given trading day:

1. If the opening price is lower than the closing or the last traded price, you can easily infer that the stock price has gone up over the trading day.
2. Conversely, you know that the stock price has declined over the trading day.
3. The price range is the highest price less the lowest price of the day. It tells you the level of volatility over the trading day.

There are several ways to visualize the open-high-low-close time series. One of the popular methods is called the Japanese candlesticks. This data visualization toolkit was widely said to be invented by Homma Munehisa (本間宗久, 1724–1803), a Japanese rice merchant and trader in the 18-th century Edo Era (江戸時代). From the candlestick patterns, stories were told that Homma could forecast the likely future direction of rice prices, with a high degree of probability—and profitability, estimated to be more than an equivalent of one trillion yen.

The up and down candles are illustration in Figure 1.2. In the Western culture, red

Figure 1.2 Data Visualization by Japanese Candlesticks.



has the connotation of danger and thus it is the color for the price going down from the opening to the closing hours. By contrast, green is the color chosen for the day when the price goes up from the opening to the closing hours. However, in the Eastern culture, it is the exact opposite, as red is the color of good luck or fortune.

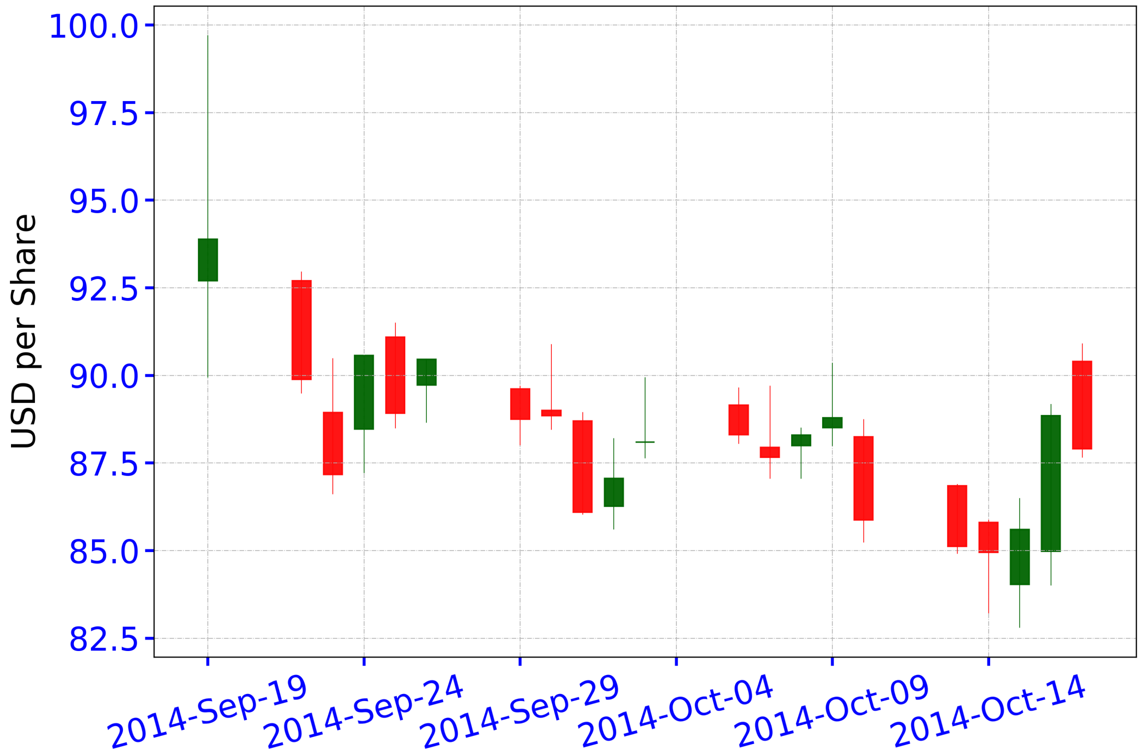
You probably notice that there are two lines coming out of each candle in Figure 1.2. The line from the body to the highest price of the day is called the upper shadow, and the line from the body to the lowest price of the day is called the lower shadow.

EXAMPLE 1.3

We plot the candlestick chart of Alibaba prices for the first day and the following four consecutive weeks of trading in Figure 1.3. You probably notice that the upper and lower shadows are very long for the first candle compared to the other 20 candles. This is a reflection of the euphoric mood and high level of speculation in the market, as the IPO is a "blockbuster" success. The wide range of about $10 tells you that trading was volatile, as valuation and re-valuation of Alibaba stock went on very rapidly on September 19, 2014.

There are 12 red candlesticks compared to 9 green ones in the chart, indicating that market players began to realize that Alibaba might be over-valued at that point of time, as the frenzy on the first trading day subsided. ■

Figure 1.3 Japanese Candlesticks of Alibaba Group Holding Ltd for the First Month After IPO.



As a quick summary, every candlestick gives you a richer set of information. You can find out the price direction by the candlestick color, opening and closing prices by the candle body, and trading range by the upper shadow less the lower shadow.

1.3 Simple Returns

Earlier we alluded to the fact that the IPO price of Alibaba is $68. Suppose you subscribe to Alibaba IPO and are lucky to receive an allotment of, say 100 shares. Moreover, suppose you sell at the closing hours of the exchange. Then your profit in dollars can be calculated by a general formula:

$$\text{Profit and Loss} = \text{Selling price} - \text{Buying price}.$$

This general formula has its basis on the cash flow analysis. When you buy and asset, you have a cash flow out, as your money is exchanged for a piece of financial contract such as stock. The cash flow is an outflow and therefore, on your P&L statement, there is a debit, which is why you have to subtract the buying price on the per share basis. On the other hand, when you sell the security, you receive cash and so there is a cash flow in.

Note that the P&L in this simple setup is the same as price change if the assumption is that you buy first to own the security at time $t-1$, and a day later you sell at time t.

Definition 1.3 **Price change** at time t of a price series $P_t, t = 1, 2, \ldots, T$ is a time series given by the price differences of any pair of adjacent prices.

$$\Delta P_t := P_t - P_{t-1},$$

for $t = 1, 2, 3, \ldots, T$, where T is the last observed value in the sample. The IPO price of the stock is denoted by P_0.

In the earlier Alibaba example, the price change on day 1 is

$$\Delta P_1 = P_1 - P_0 = \$93.89 - \$68 = \$25.89.$$

Given that you are allotted 100 shares, your P&L will be $2,589, before costs (broker commission, clearing fee, etc) and taxes.

Now, suppose you want to compare the price change across different investments, you need to define the simple return:

Definition 1.4 **Simple return**, denoted by R_t, of a price series $P_t, t = 1, 2, \ldots, T$ is a time series given by the price differences of any pair of adjacent prices divided by P_{t-1}.

$$R_t := \frac{P_t - P_{t-1}}{P_{t-1}},$$

for $t = 1, 2, 3, \ldots, T$, where T is the last observed value in the sample. ■

You may wonder why in the definition of simple return, the price change ΔP_t is divided by P_{t-1} and not P_t. To answer this question, we go back to the P&L narration in Definition 1.3. You know that the price change ΔP_t is the P&L. The buying price P_{t-1} is the money you put on the table to bet that the stock price will go up. Obviously, you need the capital to generate the profit and it is natural for you to contemplate the profit over the capital. In this context, P_{t-1} is the capital and the simple return just defined above is indeed the **return on capital**. Therefore, in the computation of simple return, you divide by P_{t-1}. In the example of the first trading day of Alibaba, where the stock price direction is in your favor, your simple return over one day is therefore your profit divided by the IPO price, namely, $25.90/$68 = 38.07%.

Note that the simple return can be re-expressed as

$$R_t = \frac{P_t}{P_{t-1}} - 1. \tag{1.1}$$

Definition 1.5 **payoff ratio** is defined as $\frac{P_t}{P_{t-1}}$. It indicates, on the per dollar of capital basis, the amount an investor will win or lose in the investment. ■

Again, in the example of Alibaba IPO, the payoff ratio is \$93.89/\$68 = 1.38. What it means is that for every \$1 of capital, it has appreciated to \$1.38. Of course, it is never guaranteed that the stock price will move up when you buy. If the payoff ratio is say, 0.70, then your every dollar invested has turned into only 70 cents, which is the same as saying that your capital has depreciated by 30%.

1.4 Log Return

The simple return defined earlier has a lower bound: -100%. This is because for any standard asset, be it stock, forex, spot commodity, or bond, at worst you can lose is your entire capital but you are under no obligation to cough out additional cash or capital to support the asset whatsoever. The worse-case scenario happens when the asset price plunges to zero; the asset becomes totally worthless. We express this idea as a lower bound of R_t, when P_t becomes zero.

$$R_t \geqq -1.$$

For some applications, the lower bound could be a hindrance. To overcome this problem, we start with the payoff ratio $\frac{P_t}{P_{t-1}}$, Which is never negative as $P_t \geqq 0$. We then consider the natural logarithm of the payoff ratio. By the property of logarithm that turns a division into subtraction, we obtain the definition of log return.

Definition 1.6 **Log return**, denoted by r_t, of a price series $P_t, t = 1, 2, \ldots, T$ is a time series given by the differences of adjacent log prices. That is

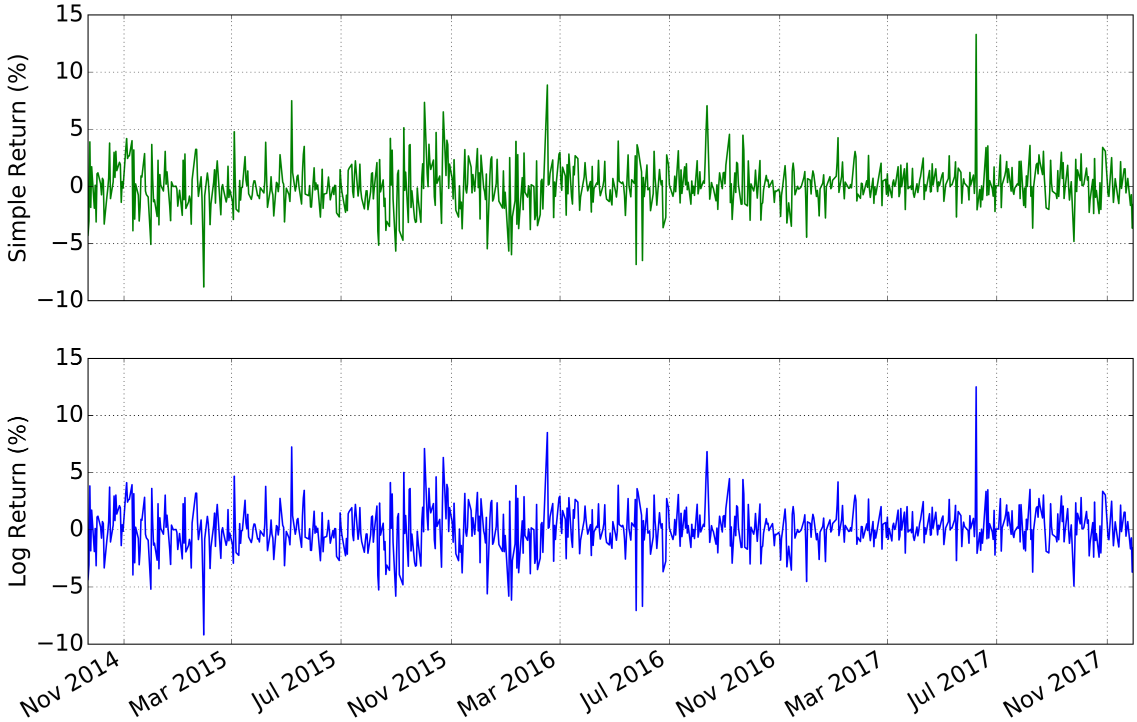
$$r_t := \ln P_t - \ln P_{t-1},$$

for $t = 1, 2, 3, \ldots, T$, where T is the last observed value in the sample. ■

Since $r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$, i.e., the log return being a logarithm function, it can become very negative when P_t is a very small number, In Figure 1.4, we plot the time series of simple and log returns of Alibaba stock. As expected, both simple and log returns are very different from the stock price series plotted in Figure 1.1.

These two time series are visually almost indistinguishable from each other. Given their definitions, there should be a relationship between the simple return R_t and the

Figure 1.4 Simple Return and Log Return of Alibaba.



corresponding log return r_t. In fact, we find that, with Equation (1.1),

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln(1+R_t). \tag{1.2}$$

We note that $|R_t| \ll 1$, i.e., the absolute value of the simple return is much smaller than 1. We perform Taylor's expansion of $\ln(1+R_t)$ and obtain

$$r_t = \ln(1+R_t) = R_t - \frac{1}{2}R_t^2 + O\big(R_t^3\big).$$

With the simple return being small, even the second-order term can be ignored, resulting in $r_t \approx R_t$. This simple mathematics allows you to understand the close similarity between the simple return and the log return in Figure 1.4.

The Taylor expansion in Equation (1.2) also shows that the log return is always smaller than the simple return. In general, we know that a log function $\ln(1+x)$, being a concave function is always smaller than the linear function x, for all x except at a special point $x = 0$ where they are equal.

In view of the apparently random nature of returns in Figure 1.4, it is natural to consider a simple mode to ascribe randomness to the asset price.

Definition 1.7 Let the payoff ratio M_t be a strictly positive random variable at time t. For emphasis, we write $M_t > 0$ for all t. Consider a time series of M_t. A **model of asset prices** P_t is as folllows:

$$P_t = P_{t-1} M_t.$$

Equivalently, we have a **model of random logarithmic asset prices**:

$$\ln P_t = \ln P_{t-1} + \ln M_t,$$

for $t = 1, 2, \ldots, T$. ■

Definition 1.7 is a simple statement claiming that the log return is random:

$$r_t = \ln P_t - \ln P_{t-1} = \ln M_t.$$

Let $\xi_t = \ln M_t$, which is to say, $r_t = \xi_t$ is random.

1.5 Multi-Period Returns

So far, in the definitions of simple return and log return, we have implicitly assumed that the two prices are adjacent chronologically. In other words, the time interval is one unit or one period. To endow the model of random log prices with a richer structure, we need to consider multi-periods. That is, in general, we consider P_t versus P_{t-q} for a given integer $q \geqq 1$. For example, $q = 1$ is the daily log return, $q = 2$ represents bi-daily log return, $q = 3$ corresponds to tri-daily log return, and in general, we speak of q-daily log return.

An interesting property of the payoff ratio in the context of multi-period return is called the rule of telescopic multiplication:

$$\frac{P_t}{P_{t-q}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \frac{P_{t-2}}{P_{t-3}} \times \cdots \times \frac{P_{t-q+2}}{P_{t-q+1}} \times \frac{P_{t-q+1}}{P_{t-q}}. \tag{1.3}$$

When you apply the natural logarithm on both sides, you obtain

$$r_{q,t} = r_t + r_{t-1} + r_{t-2} + \cdots + r_{t-q+2} + r_{t-q+1},$$

where $r_{q,t}$ is a notation for q-daily return:

$$r_{q,t} := \ln\left(\frac{P_t}{P_{t-q}}\right) = \ln P_t - \ln P_{t-q}.$$

Therefore, q-daily log return is a sum of q daily returns.

The exponential function is the reverse function of logarithm, i.e., $\exp\big(\ln(x)\big) = x$. Another property of multi-period log return is that

$$\exp\left(r_{q,t}\right) = \frac{P_t}{P_{t-q}},$$

which is, by definition,

$$P_t = P_{t-q} \exp\left(r_{q,t}\right) = P_{t-q} \exp\left(r_t + r_{t-1} + r_{t-2} + \cdots + r_{t-q+2} + r_{t-q+1}\right).$$

From time $t-q+1$ to t, there are q periods. We write the arithmetic average log return as

$$\bar{r} := \frac{1}{q}\left(r_t + r_{t-1} + r_{t-2} + \cdots + r_{t-q+2} + r_{t-q+1}\right).$$

It follows that

$$P_t = P_{t-q} \exp\left(q\bar{r}\right).$$

In other words, the average log return $\bar{r}$ is the **continuously compounding return**, and q is the length of the holding period. Even so, daily log return is continuously compounding return over one period with $q = 1$.

Now, from the institutional investment perspective, one of the greatest concerns of any fund manager of a portfolio is the asset under management (AUM). A more relevant return to fund managers is the notion of geometric average over a number of years.

Definition 1.8 **Geometric average return**, denoted by g_t is primarily used for calculating the average rate per period on investments that are compounded over multiple periods. It is defined with respect to the payoff ratio:

$$g_t := \left(\frac{P_t}{P_{t-q}}\right)^{\frac{1}{q}} - 1. \tag{1.4}$$

■

By the rule of telescopic multiplication, Equation (1.3) we can rewrite g_t as

$$1 + g_t = \left(\frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \frac{P_{t-2}}{P_{t-3}} \times \cdots \times \frac{P_{t-q+2}}{P_{t-q+1}} \times \frac{P_{t-q+1}}{P_{t-q}}\right)^{\frac{1}{q}}$$

Each period's payoff ratio is related to the simple return, i.e., $\frac{P_{t-i}}{P_{t-i-1}} = 1 + R_{t-i}$, and $i = 0, 1, 2, \ldots, q$. Consequently,

$$(1 + g_t)^q = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-q+1}). \tag{1.5}$$

In this form, you can easily obtain an insight into the average nature of g_t. The simple returns most likely differ from one period to another, but the geometric average return g_t is a single number that tells you the average growth per periods for q periods. Intuitively, what you find is that every dollar invested will become $(1 + g_t)^q$ if you hold on to the investment.

To obtain further insight into g_t, we take logarithm on both sides of Equation (1.5) to obtain

$$q \ln(1 + g_t) = \ln(1 + R_t) + \ln(1 + R_{t-1}) + \cdots + \ln(1 + R_{t-a+1}).$$

Noting the relationship between the log return and the simple return, Equation (1.2), we have

$$\ln(1+g_t) = \frac{r_t + r_{t-1} + \cdots + r_{t-q+1}}{q} = \bar{r}, \tag{1.6}$$

which leads to

$$g_t = e^{\bar{r}} - 1 \approx \bar{r} + \frac{1}{2}\bar{r}^2 + O\left(\bar{r}^3\right).$$

Thus, we see that the geometric average return g_t is always larger than the arithmetic average of the log returns. Even so, if $q = 1$, then $\bar{r} = r_t$, and we have $g_t \geqq r_t$. They are equal in the trivial case when both returns are 0.

1.6 Time-Weighted Return

In the investment industry, investors at times may invest more by infusion of more money. At times, investors may invest less by withdrawing money from their investment accounts. How would you, as the fund manager, compute some sort of average return for the investors?

The answer to this question lies in the calculation of simple returns followed by computing the geometric average return. The resulting average is referred to as the time-weighted return. Perhaps the best way to explain time-weighted return procedure is by examples.

EXAMPLE 1.4

An institutional investor, Rotsevni, invests \$1 million into your fund on December 31. Rotsevni is your only client. Ten months later on October 31 the following year, through your tactical and strategic strategies, the value of the portfolio becomes \$1.2 million. On that day, Rotsevni invests \$0.8 million more on October 31, bringing the asset under management to \$2 million. By the end of the year, the portfolio value becomes \$1.9 million because a particular blue-chip stock in your portfolio is in trouble, and its share price plunges. You need to report to your client, and the obvious question is, "what's the return?"

For the first 10 months, the simple return is

$$\frac{1.2 - 1.0}{1.0} = 20\%.$$

For the next two months, the simple return is

$$\frac{1.9 - 2}{2} = -5\%.$$

Having computed the simple returns, you then proceed to compute the annual return for your client by the geometric average

$$(1+0.20)(1-0.05) = 1.14.$$

Therefore, the time-weighted return is $(1.14-1) = 14\%$. ■

EXAMPLE 1.5

Suppose you are investing on behalf of your only client as in Example 1.4. Again, the portfolio grows by 20% over the first 10 months. Instead of injecting more fund, your client withdraws \$0.2 million, bringing the AUM to \$1.2 million – \$0.2 million = \$1 million, as of October 31. Likewise, your portfolio takes a knock and its value becomes \$0.95 million by the end of the year.

For the first 10 months, the return is 20% as before. For the last two months of the year, the simple return is

$$(0.95-1)/1 = -5\%.$$

The geometric average return is again $(1+0.2)(1-0.05)-1 = 14\%$. ■

Actually, time-weighted return may be a misnomer. One could easily fall into the trap of interpreting it literally as an average weighted by time. In the two examples, the first period is longer and the second period is on 2 months. So you may be tempted to compute the following time-weight average:

$$20\% \times \frac{10}{12} + (-5\%) \times \frac{2}{12} = 15.83\%,$$

which is higher than the geometric average return of 14%.

Therefore, it is very important to know that time weighted return is essentially the geometric average return that ought to be utilized to find the average of simple returns in the multi-period context.

As a matter of fact, in portfolio management, when your investor wants to invest more or request for a withdrawal, it is as if you have to reset the investment. Specifically, one paper, you sell the portfolio and the current price so as to compute the simple return, which yields 20% in the two examples above. Then, on paper, you buy back the original portfolio at the current price if the investor pumps in more money. You have to actually buy the asset with the additional money at the current price. If the investor requests for a withdrawal, you have to actually sell the asset in such a way that the proceeds equal the amount of withdrawal.

1.7 Case Study: GIC

In 1981, Mr Goh Keng Swee, then Chairman of the Monetary Authority of Singapore, saw the trend of Singapore's growing foreign reserves in the midst of heightened inflation risk. Being also the first Deputy Prime Minister, he started the initiative to set up the Government of Singapore Investment Corporation Pte Ltd (GIC), with the mandate to invest Singapore's foreign reserves, so as to earn reasonable returns within acceptable risk limits over the long term.

GIC is one of the so-called sovereign wealth funds in the world. As the name suggests, a sovereign wealth fund is a state-owned investment vehicle to manage national budget surpluses, accumulated over the years due to favorable macroeconomic, trade, and fiscal positions, coupled with long-term budget planning and spending restraint. Traditionally, sovereign wealth funds prefer to remain low key and opaque as they are under no obligation to disclose their financial positions. In fact, for whatever reasons, the states forbid their sovereign wealth funds to disclose information that might compromise their positions.

With some quarters expressing concerns that sovereign wealth funds might destabilize markets and financial systems—especially those cross-border investments—IMF and OECD were called upon to develop a non binding code of conduct for the funds to agree to operate under. The intention is to bring about some financial stability in the turbulent market of 2008. GIC, being a sovereign wealth fund, answers only to one and only one client: the Ministry of Finance. Nevertheless, GIC participated actively in discussions on the codes of investment practices and principles for sovereign wealth funds to abide by voluntarily.

Against this backdrop, in 2008, GIC voluntarily published for the first time an annual report containing information on its 20-year returns, as well as the people who were leading this extraordinary private limited. Notably, Robert Litterman was among the advisers to the GIC Board of directors. He and Fischer Black had developed the well known Black-Litterman model [BL92] for optimizing the return of a portfolio while taming the risk. Overall, GIC has strong industry connections to attract and retain top talents all over the world.

Recall that the mission of GIC is to preserve and enhance the international purchasing power of the reserves. Therefore, the effects of global inflation have to be taken into account when computing the portfolio return. Essentially, inflation is about the increasing trends in the price levels of goods and services. Inflation erodes your purchasing power; to buy the same item in the future, you need to pay more dollars as opposed to buying a unit of the item now. In other words, the amount of goods and services you can buy today is less than what you could have bought in the past if your wealth is not growing.

Definition 1.9 **Nominal return** is the return that does not take inflation effects into account. **Real return**, on the other hand, is the return adjusted for changes in price levels due to inflation. ■

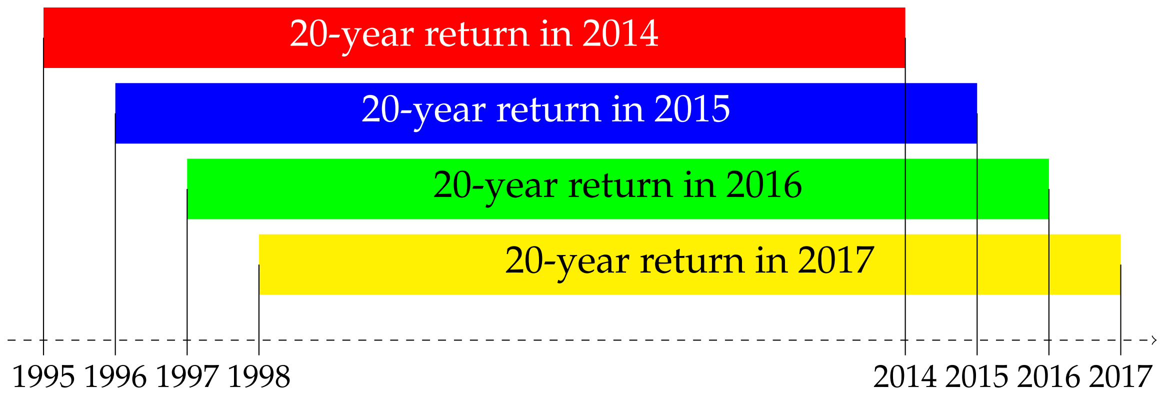
In its annual report for 2010, GIC published a chart that plots the nominal returns in US dollars and real returns for 2001 to 2007 as well. By carefully reading off the chart, you can estimate the nominal returns in US dollars for these years before 2008[1]. In Table 1.2, returns shaded in light green are estimated from the chart in 2010 annual report, while those shaded in light blue are the exact numbers captured from the annual reports. It is important to mention that the real return is independent of the currency because in the adjustment for inflation, the inflation rates used for adjustments must be of the same currency with which to compute the nominal returns. Hence, the real return for 2008 is the reported number and thus shaded in blue.

Table 1.2 GIC's 20-Year Annual Returns in Percent.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Nominal	9.5	9.5	8.7	9.5	9.7	8.5	7.9	7.7	5.7	7.1	7.2	6.8	6.5	6.5	6.1	5.7	5.7
Real	5.8	5.8	4.5	5.0	4.9	4.9	4.8	4.5	2.6	3.8	3.9	3.9	4.0	4.1	4.9	4.0	3.7

All the figures in Table 1.2 are annual time-weighted returns over 20 years. Figure 1.5 illustrates the notion of rolling window of one year implicit in Table 1.2. As an example, the nominal geometric average return of 5.7% in the 2017 annual report is the return obtained from comparing the GIC portfolio value of March 1997 with that of March 2017.

Figure 1.5 Illustration of Rolling by One Year.



[1]In the first 2008 annual report, there is a chart showing the time series of nominal returns for 2001 to 2007 as well. But these nominal returns are in Singapore dollars. Since GIC uses US dollars as the base currency from 2009 onward, we have to use the alluded 2010 chart.

Applying Equation (1.4), we have

$$\left(\frac{P_{2017}}{P_{1997}}\right)^{\frac{1}{20}} - 1 = 5.7\%.$$

To get a more intuitive picture, we rewrite this equation as

$$P_{2017} = P_{1997}(1+0.057)^{20}.$$

To make the concept of geometric average return more concrete, suppose you had US \$100 in 1997 and suppose you could invest your money in the exact way GIC invests. Your \$100 would become $\$100 \times (1.057)^{20} = \$303.04$ in 2017. Given that the corresponding 20-year real return is 3.7%, your purchasing power in 2017 will be $\$100 \times (1.037)^{20} = \$206.81$, which is twice more than you could afford to buy 20 years ago.

What story does the two time series in Table 1.2 tell you? In general, when the return decreases from one year to the next year, it is likely that GIC had suffered losses for that year. Since reaching the peak of 9.7% in 2015, GIC had four consecutive losing years. Noticeably, the drop of nominal return from 7.7% to 5.7% in 2009 is the most drastic ever for the period 2001 through 2017. This decline is inevitable because of the global financial crisis. Though there was a V-shape recovery in 2010, the nominal return continues to trend lower to 6.1% in 2017. The same story can be said about the real return.

The takeaway is that it is not easy even for big institutional investors such as GIC to make money. Part of the reason is that there are more and more sovereign wealth funds coming into the market. Gone are the low-lying fruit in the past; the portfolio management game has become more complex in the global digital age. It would be quite a herculean task and long process for GIC to recover back to the real return of 5.8% registered in 2001.

1.8 Total Returns

Though not obligatory, companies usually pay dividends to their shareholders. Dividends are typically a part of the profit that the company shares with its shareholders. Dividends can be issued in various forms, such as cash payment, stocks, or any other form. A company's dividend is decided by its board of directors and it requires shareholders' approval.

Suppose you invest in a dividend-paying stock, which pays dividend on a regular basis. To determine whether you would get a dividend, you need to check a few important dates. When a company declares a dividend on the **declaration date**, it announces three important dates and the dividend per share. In chronological order, they are ex date, record date, and payment date.

1. **Ex date** is the cutoff date before which existing and new shareholders are entitled to receive the upcoming dividend payments.
2. **Record date** is the date at which the book containing the particulars of each shareholder such as the number of shares owned, mailing address, etc is updated and closed.
3. **Payment date** is the earliest date on which you will receive your dividend

Ex-date is very important. If you purchase a company's shares before the ex-dividend date, you are entitled to receive the upcoming dividend from the company. But if you buy on the ex-dividend date or after, you will not receive the upcoming dividend payment.

Table 1.3 shows the dividend history of Coca-Cola Company. It is clear that the ex date is one business day before the record date, and the payment date is typically two weeks after the record date.

Table 1.3 Dividend History of Coca-Cola.

Declaration Date	Type	Amount	Ex Date	Record Date	Payment Date
2017-10-19	Cash	$0.37	2017-11-30	2017-12-01	2017-12-15
2017-07-20	Cash	$0.37	2017-09-14	2017-09-15	2017-10-02
2017-04-27	Cash	$0.37	2017-06-13	2017-06-15	2017-07-03
2017-02-16	Cash	$0.37	2017-03-13	2017-03-15	2017-04-03
2016-10-20	Cash	$0.35	2016-11-29	2016-12-01	2016-12-15
2016-07-21	Cash	$0.35	2016-09-13	2016-09-15	2016-10-03
2016-04-28	Cash	$0.35	2016-06-13	2016-06-15	2016-07-01
2016-02-18	Cash	$0.35	2016-03-11	2016-03-15	2016-04-01
2015-10-15	Cash	$0.33	2015-11-27	2015-12-01	2015-12-15
2015-07-16	Cash	$0.33	2015-09-11	2015-09-15	2015-10-01
2015-04-30	Cash	$0.33	2015-06-11	2015-06-15	2015-07-01
2015-02-19	Cash	$0.33	2015-03-12	2015-03-16	2015-04-01
2014-10-16	Cash	$0.305	2014-11-26	2014-12-01	2014-12-15
2014-07-15	Cash	$0.305	2014-09-11	2014-09-15	2014-10-01
2014-04-24	Cash	$0.305	2014-06-12	2014-06-16	2014-07-01
2014-02-20	Cash	$0.305	2014-03-12	2014-03-14	2014-04-01
2013-10-17	Cash	$0.28	2013-11-27	2013-12-02	2013-12-16
2013-07-18	Cash	$0.28	2013-09-12	2013-09-16	2013-10-01

For every share, a shareholder who holds the share before the ex date is entitled to receive a dividend, which we denote it as D_t. In other words, D_t is dividend per share that investors will receive. A natural question arises: What should the date t be? Should t be the announcement date, ex date, record date, or payment date? As mentioned earlier, if you purchase the stock after the ex date, you will not receive the upcoming dividend. On the other hand, if you sell the stock on or after the ex date, you still get to receive the dividend.

Moreover, by a simple argument of no risk-free arbitrage, the stock price should drop by an amount equal to the dividend per share D_t on ex date. Suppose the stock price does not change from $t-1$ to t. Investors will always buy the stock at day $t-1$ and sell it on ex dividend day t, and they will receive the dividend without risk. Therefore, holding all the market conditions constant, the share price on ex date t will have to drop by D_t. It follows that t should be the ex date.

Now, in computing the return on an asset, often times it is important to take into account the cash flow from dividend.

Definition 1.10 **Total return**, denoted by $\breve{R}$, is the return that recognizes dividend D_t as the cash flow receipt in the P&L computation, resulting in

$$\breve{R}_t := \frac{P_t + D_t - P_{t-1}}{P_{t-1}}. \tag{1.7}$$

■

Albeit not guaranteed and uncertain dividend is nevertheless a source of income. From the investment standpoint, P_t is the current market value of the stock. Since the stock is generating income, it is a common practice to compute the yield with respect to the market value of your capital P_t.

Definition 1.11 The ratio of dividend D_t to stock price P_t is called the dividend yield.

■

Proposition 1.12 If the total return and the simple return are given for time t, then the dividend yield can be inferred by the following formula:

$$\frac{D_t}{P_t} = \frac{\breve{R}_t - R_t}{1 + R_t}. \tag{1.8}$$

■

Proof: First, we express the total return Equation (1.7) as

$$\breve{R}_t = \frac{D_t}{P_{t-1}} + \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{D_t}{P_{t-1}} + R_{t+1}.$$

Shifting R_t to the right hand side, and multiplying the dividend yield by $1 = \frac{P_t}{P_t}$, we obtain, after swapping the denominators,

$$\breve{R}_t - R_t = \frac{D_t}{P_t}\frac{P_t}{P_{t-1}} = \frac{D_t}{P_t}\left(1 + R_t\right).$$

Dividing both sides by $1+R_t$ and the proof of Equation (1.8) is complete. ∎

EXAMPLE 1.6

Suppose you can observe the simple and total returns of a stock but you do not have information about dividends. Specifically, on day t, the simple return is 1% and the total return is 1.9%. What is the (implied) dividend yield?
Appying Equation (1.8), we obtain

$$\frac{1.9\% - 1\%}{1+1\%} = \frac{0.9\%}{101\%} = 0.0089 = 0.89\%.$$

∎

The amount of 0.89% in the above example may not seem a lot. But notice from Table 1.3 that the company pays dividend quarterly. You may add up the 4 dividend yields together to arrive at the annualized dividend yield.

EXAMPLE 1.7

For the 4 dividends in 2017 in Table 1.3, the end of day stock prices of Coca Cola a day before ex dates are, respectively, \$42.03 (March 13), \$45.03 (June 13), \$46.11 (September 14), and \$45.77 (November 30). From Table 1.3, each dividend cash amount is \$0.37. Therefore, the annual dividend yield is

$$0.37 \times \left(\frac{1}{42.03} + \frac{1}{45.03} + \frac{1}{46.11} + \frac{1}{45.77}\right) = 3.31\%.$$

∎

Example 1.7 is just one of the many ways to compute annual dividend yield. A simpler approach could be simply adding up all the quarterly payments and divide by the current price. As an illustration, suppose today's date is June 30, 2017. A backward-looking dividend yield is to take 4 most recent dividend payments before June 30, namely, two dividends of \$0.35 each in the second half of 2016, and two dividends of \$0.37 each in the first half of 2017. Given that the stock price of Coca-Cola company is \$44.85 on June 30, 2017, the dividend yield is obtained as 2×(\$0.35 + \$0.37)/\$44.85 = 3.21%. An implicit assumption in this approach of computing the dividend yield is that investors are holding the stock for at least a year.

1.9 Dividend Adjustment

How should we adjust stock prices in order to take into account dividend payments? There are at least two reasons why we want to adjust stock prices. First and foremost, it is at times imperative to analyze total return, taking into account dividend reinvestments for reporting performance and so on. Second and equally important, we may need to apply trading strategies based on the stock prices series.

Suppose you receive the dividend D_t and you immediately reinvest this D_t into the same stock. Suppose you initially have N shares. The total dividend amount you receive in dollars is ND_t. With this amount of cash, you can buy $\frac{ND_t}{P_t}$ shares. You have just transformed the cash dividend into shares. So at the end of time t, your number of shares has increased from N to $N\left(1+\frac{D_t}{P_t}\right)$. Suppose you can *hypothetically* liquidate your entire position. Let us calculate your return on paper:

$$\check{R}_t = \frac{N\left(1+\frac{D_t}{P_t}\right)P_t - NP_{t-1}}{NP_{t-1}} \tag{1.9}$$

$$= \frac{P_t + D_t - P_{t-1}}{P_{t-1}} \tag{1.10}$$

$$= \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}. \tag{1.11}$$

The second equality (1.10) is exactly the same as our earlier definition of total return, Equation (1.7).

Clearly, there are two return components as you might expect by looking at Equation (1.11). The first component is the simple return $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$, which captures capital appreciation ($R_t > 0$) or depreciation ($R_t < 0$). The second component is due to dividend "reinvestment", which is never negative.

In reality, of course, you do not receive dividend cash on ex date t. What you can do, nevertheless, is to borrow money equivalent to ND_t, and use that amount of cash to reinvest, i.e., transform cash receipt from owning into more shares on the same stock. Since you are entitled to receive ND_t on the date of payment, you are able to repay the bank[2].

1.9.1 Backward Adjustment

The goal we have in mind is to adjust stock prices to account for dividends.

[2] Obviously, you need to pay interest to the lending bank. In all the definitions, we are not taking all the transaction costs into account. We also ignore the interest paid. So you can expect the actualized total return to be smaller than the total return on paper.

Definition 1.13 Knowing the ex date t and the dividend per share D_t, the **dividend adjustment factor** is defined as

$$B_t := \frac{1}{1 + \frac{D_t}{P_t}}.$$

The **adjusted stock price** for $s = t-1, t-2, \ldots 2, 1, 0$ is defined as

$$P_{b,s} = P_s B_t.$$

■

Proposition 1.14 The total return can be expressed in terms of the adjusted price as follows:

$$\breve{R}_t = \frac{P_t - P_{b,t-1}}{P_{b,t-1}}. \tag{1.12}$$

It has the same form of a simple return. ■

Proof: We multiply the total return Equation (1.9) by $1 = B_t/B_t$ to obtain

$$\breve{R}_t = \frac{NP_t - NP_{b,t-1}}{NP_{b,t-1}} = \frac{NP_t - NP_{t-1}B_t}{NP_{t-1}B_t} = \frac{P_t - P_{b,t-1}}{P_{b,t-1}}.$$

In other words, to compute total return, we need to use the time series of adjusted prices. ■

Moreover, if neither s nor $s-1$ is an ex date, it can be easily shown that

$$\breve{R}_s = \frac{P_{b,s} - P_{b,s-1}}{P_{b,s-1}} = \frac{P_s B_t - P_{s-1}B_t}{P_{s-1}B_t} = \frac{P_s - P_{s-1}}{P_{s-1}} = R_s.$$

This result is consistent with Equation (1.8). Since $D_s = 0$, it must be that $\breve{R}_s = R_s$. Also, it is important to emphasize that for any arbitrary non-zero number α,

$$R_t = \frac{\alpha P_t - \alpha P_{t-1}}{\alpha P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

In other words, using adjusted prices on days that do not involve a dividend payment will produce the same value for simple return as using the unadjusted prices. It also follows that the log return will not be affected by the adjustment factor when there is no dividend payment.

As in Table 1.3, a blue chip company such as Coca Cola pays dividend on a regular basis. For each dividend, there will be a dividend adjustment factor. Suppose you have information about all the ex dates t_i and the dividend per share D_{t_i}, for $i = 1, 2, \ldots, n$, where n is the latest distribution of dividend. Obviously, you have all the stock prices P_t, $t = 0, 1, 2, \ldots, T$, that you want to adjust. We label T as the latest or most current time. Stock prices on the ex dates are indicated by P_{t_i}. The algorithm for adjusting the stock prices backward is as follows:

1. Compute all the dividend adjustment factors B_{t_i}, $i = 1, 2, \ldots, n$.
2. For all the oldest prices before the first ex date t_1, multiply them by B_{t_1}.
3. For all the oldest prices before t_2, multiply them by B_{t_2}.
4. Do likewise for $i = 3, 4, \ldots, n$.
5. For the most recent prices from t_n onward, no adjustment is needed.

The outcome is that the prices before t_1 are multiplied by all the dividend adjustment factors, i.e.,

$$P_{b,s} = B_{t_1} \times B_{t_2} \times \cdots \times B_{t_n} \times P_s, \quad \text{for } s = 0, 1, 2, \ldots, t_1 - 1.$$

For stock prices between t_1 and $t_2 - 1$, they are adjusted as

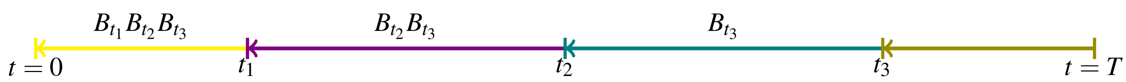
$$P_{b,s} = B_{t_2} \times B_{t_3} \times \cdots \times B_{t_n} \times P_s, \quad \text{for } s = t_1, t_1 + 1, t_1 + 2, \ldots, t_2 - 1.$$

In general, with $t_0 = 0$, and for $j = 1, 2, \ldots, n$, the adjusted prices are given by

$$P_{b,s} = B_{t_j} \times B_{t_{j+1}} \times \cdots \times B_{t_n} \times P_s, \quad \text{for } s = t_{j-1}, t_{j-1} + 1, t_{j-1} + 2, \ldots, t_j - 1.$$

The algorithm for backward adjustment is illustrated in Figure 1.6.

Figure 1.6 Illustration of Backward Dividend Adjustments.



Since $B_{t_i} < 1$, past historical prices will become smaller and smaller. This backward adjustment method is popularly employed by most financial service providers. The main merit is that the most recent prices are the same as what you observe from stock exchanges on which the stocks are traded.

1.9.2 Forward Adjustment

But as investors, especially those long-term ones such as GIC, you ask the most important question: If I invest \$1,000 today, what is a reasonable estimate of the total value (before costs) of my investment in the future, at least on paper? To answer this question, you will adjust the stock prices forward instead. In so doing, you are constructing a time series of total-return stock prices.

Looking at Equation (1.9), we need to multiply $P_t, P_{t+1}, P_{t+2}, \ldots$ by $1 + \frac{D_t}{P_t}$, which is the inverse of B_t in Definition 1.13.

Definition 1.15 The **forward dividend adjustment factor** F_t is defined as

$$F_t := \frac{1}{B_t} = 1 + \frac{D_t}{P_t},$$

where t is the ex date. The **total-return stock prices** are given by

$$P_{f,s} := P_s F_t, \quad \text{for } s = t, t+1, t+2, \ldots.$$

■

Suppose you have a series of stock prices $P_t, = 0, 1, 2, \ldots, T$, where T is the latest or most current time. The algorithm for forward adjusting the stock prices is described as follows:

1. Calculate all the forward dividend adjustment factors F_{t_i}, $i = 1, 2, \ldots, n$.
2. Start from chronologically the oldest date, i.e., $t = 0$.
3. Do not adjust the stock prices before the first ex date t_1.
4. Multiply by F_{t_1} all prices from P_{t_1} to P_T.
5. Multiply by F_{t_2} all prices from P_{t_2} to P_T.
6. Do likewise for $i = 3, 4, \ldots, n$.
7. No forward adjustment is needed before t_1.

The outcome from forward adjustment is that from the most current ex date t_n to the most current time T, the stock prices within and inclusive of these two dates get adjusted by all the forward dividend adjustment factors F_{t_i}. In other words,

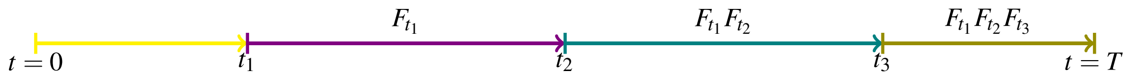
$$P_{f,s} = \prod_{i=1}^{n} F_{t_i} P_s, \quad \text{for } s = t_n, t_n + 1, \ldots, T.$$

Generally, when $k < n$,

$$P_{f,s} = \prod_{i=1}^{k} F_{t_i} P_s, \quad \text{for } s = t_k, t_{k-1} + 1, \ldots, t_{k+1} - 1.$$

The algorithm is illustrated in Figure 1.7.

Figure 1.7 Illustration of Forward Dividend Adjustments.



EXAMPLE 1.8

We download the stock prices of Coca Cola from **Yahoo! Finance**. With reference to Table 1.3, we set our sample period starting from September 3, 2013 through January 2 ,2018. The results of forward dividend adjustments are plotted in Figure 1.8. Clearly, the total-return price series starts to become larger and larger compared to unadjusted price series as time increases. If you have bought 100 shares on September 3, 2013 at the price of \$37.90 per share, the reinvestment will grow, in a compounded fashion, the number of shares to 115.19 shares at the end of the sample period. In terms of returns, over the sample period,

$$\text{Price return} = \frac{45.54 - 37.90}{37.90} = 20.16\%$$
$$\text{Total return} = \frac{52.46 - 37.90}{37.90} = 38.41\%$$

The total return is about 18.25% higher than the price return without reinvestment.

■

1.9.3 Yahoo! Finance Method

Many finance oriented bloggers and web sites rely on Yahoo! Finance as their data source, as Yahoo! Finance provides historical prices of stocks for free. Each row of the time series of historical prices has the data fields of Date, Open, High, Low, Close, Adj Close and Volume. The column of Adj Close corresponds to the Close prices adjusted for dividends and stock splits. The adjustment is backward and hence the most current Adj Close and the Close price are no different.

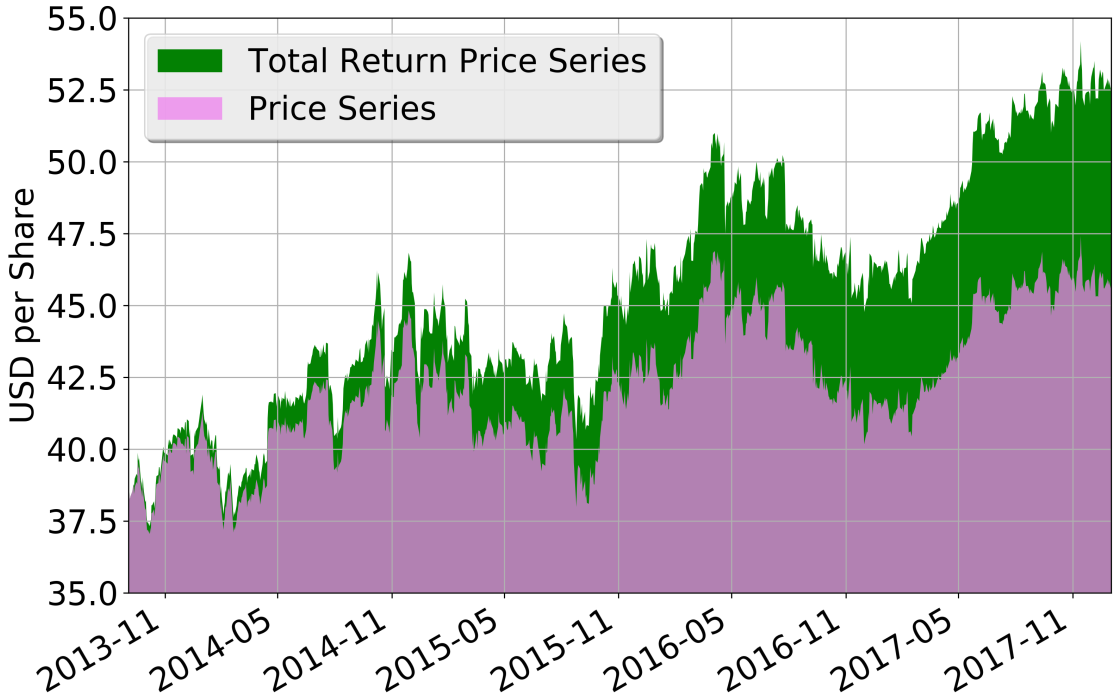
How does Yahoo! Finance perform backward adjustment for dividends? As a quant, you need to perform a little bit of "reverse engineering" to figure out.

It appears that Yahoo! Finance uses the following formula to calculate the backward adjustment factors Y_t:

$$Y_t := 1 - \frac{D_t}{P_{t-1}} = \frac{P_{t-1} - D_t}{P_{t-1}} \leqq 1. \tag{1.13}$$

This formula for dividend adjustment assumes that the dividend reinvestment is done at the price of P_{t-1}–D_t. An implicit assumption is that you could reinvest the dividend at the closing price less the upcoming dividend per share (D_t) a day prior to the ex date, i.e., $t-1$.

Figure 1.8 Coca Cola Prices and Total-Return Prices.



Obviously, this assumption is very difficult to fulfill simply because on any given day s, to trade at the price closing price minus a certain amount is almost impossible because P_{t-1} is the *closing* price. The implication is that the total return computed by the adjustment with Equation (1.13) may not be accurate. Be that as it may, if the intent is to adjust for stock prices and to be compatible with the historical prices, say, half a century ago, then the adjustment by Yahoo! Finance is as good, or as bad as the methods discussed earlier.

1.10 Summary

EXERCISES

1.1 The one-year returns of a portfolio are 2.22%, -7.77% , and 3.33%. What is the geometric average return of the portfolio?

1.2 The arithmetic monthly average log return of a portfolio over 20 months is 1.717%.

a) What is the value of the average monthly geometric return?

b) What is the value of the average annualized geometric return?

1.3 At year t, the two-year geometric average return of a portfolio is 5.55%, and the three-year geometric average return is -8.88%. What is the one-year geometric average return for year $t-2$?

1.4 You are a long term investor and you invest \$1,000 in 2000 and after 10 years, your investment value is \$4,000. What is the average geometric return?

1.5 The dividend yield at day t of an ex date is 0.5%, and the simple return is -1%.

a) What is the value of the backward dividend adjustment factor?

b) What is the value of the backward dividend adjustment factor using the method of Yahoo! Finance?

1.6 An institutional investor, Creaj, invests \$100 billion into your fund on December 31. 3 months later on March 31 the following year, the value of the portfolio becomes \$103 billion. On that day, Creaj invests \$1 billion more. By the end of the year, the portfolio value becomes \$110 billion, and Creaj withdraws \$2 billion from the fund. What is the 1-year return for Creaj?

1.7 The backward adjustment formula of Yahoo! Finance can be written as a function of dividend yield at time t and the simple return. Give a proof of this claim.

1.8 Write a summary of at least 153 words on what you have really learned from this chapter on the subject of returns.

REFERENCES

[BL92] Fischer Black and Robert Litterman, *Global portfolio optimization*, Financial Analysts Journal **48** (1992), no. 5, 28–43.