
ALGORITHMIC FINANCE

Christopher Ting
Lee Kong Chian School of Business
Singapore Management University

Version 0.1
June 9, 2018

CONTENTS

Indexes	5
2.1 A Brief History	5
2.2 Index Weighted by Price	8
2.2.1 Four Dow Jones Average Indexes	8
2.2.2 Nikkei 225 Index	10
2.2.3 How to Construct a Price-Weighted ETF?	12
2.3 Index Weighted by Market Capitalization	14
2.3.1 Value-Weighted Index	14
2.3.2 How to Construct a Value-Weighted ETF?	17
2.3.3 Free Float	20
2.4 Case Study: Hang Seng Index	20
2.5 Equally Weighted Index	22
2.5.1 Example: Value Line Index	23
2.5.2 How to Create an Equally Weighted ETF?	23
2.5.3 Value-Weighted versus Equally-Weighted ETFs	24
2.6 Re-balancing	26
2.6.1 Price-Weighted Index	26
2.6.2 Value-Weighted Index	27
2.6.3 Equally Weighted Index	27
2.6.4 Section Summary	29
2.7 Reconstitution	29
2.7.1 Price-weighted index	30
2.7.2 Value-Weighted Index	31
2.7.3 Equally weighted index	34
2.8 Summary	35
Exercises	35

CHAPTER 2

INDEXES

For those who have made any attempt to penetrate their mysteries, index numbers seem to have a perennial fascination.

Irving Fisher (1922)

An index is a system of numbers for comparing values of interest to market participants. In his bid to demystify the notion of index, Fisher (1922) suggests that it is essentially some sort of an average of prices and quantities. Stock market indexes first appeared in the 19th century. With the benefit of “now sight”, stock market indexes arguably can be regarded as one of the earliest financial innovations, which is far ahead of its time. Today, many exchange traded funds (ETF) are being created to track their respective indexes as closely as possible. An ETF is essentially a basket of securities designed specifically to mimic the index behaviors, as closely as possible. Without the indexes, you would not have ETFs to invest in.

2.1 A Brief History

In the United States, many companies have their stocks listed and traded on the stock exchanges. Obviously, it is difficult for investors to look at *all* the prices of publicly listed stocks at any given time, especially on the continuous basis. Yet, market participants need to know whether the stock market as a whole is going up or down. In the

era when neither computers nor information display systems were present, it was difficult for a layman to collect and keep track of stock prices. This need was identified and filled by 3 journalists: Charles Dow, Edward Jones, and Charles Bergstresser¹. In November 1882, they founded the Dow Jones & Company as a financial news provider. It became the delivery platform for Dow's invention of a stock market index, as a single number, that tracks the stock market direction and movement over time.

The first ever U.S. index could be traced back to July 3, 1884. Dow Jones & Company started to publish the average price of the prices of 9 railroads and 2 industrial companies in its *The Customers' Afternoon Letter*, which later was branded *The Wall Street Journal* by Bergstresser. This average index is a precursor to what is known as the Dow Jones Transportation Average (DJTA) index today. The 20-stock version was introduced on September 23, 1889. It had 18 railroad stocks and 2 industrial stocks. Dow created these averages to illustrate his theories in what is today called technical analysis (see [Lo 2016](#)).

In the beginning of the 19th century, the U.S. economy was still in a developing, pre-industrial stage. In the later half of the century, many industrial products and services, for example, electricity and telegraph for communication, became more and more ubiquitous. Yet, shares issued by industrial companies such as General Electric were considered to be highly risky. Nevertheless, on May 26, 1896, the Dow Jones Industrial Average (DJIA) index consisting of a dozen component stocks, was officially launched. Interestingly, DJTA, DJIA, and the Dow Jones Utility Average (DJUA) index launched in 1929 are still being published and used by investors today. This is quite a remarkable feat, considering the fact that there were many other indexes in the early half of the 20th century (see [Cowles 3rd 1939](#)).

The rival index provider, Standard & Poor's (S&P), debuted their equity indexes in 1923, covering 233 stocks in 26 sectors on a weekly basis. The daily indexes were introduced in 1928 and consisted of a 90-stock average including 50 industrials, 20 rails and 20 utilities. On March 4, 1957, S&P expanded the coverage to 500 stocks and renamed it the S&P 500 index. Its constituents are 500 leading US company stocks.

Notably, the method by which the S&P 500 index is computed is different from the price-weighted method used by Dow Jones. For the first time in history, S&P uses the market capitalizations rather than prices only to construct the index. The market capitalization of a company reflects the valuation of the company's equity through the market mechanism. The S&P 500's approach to constructing an index is considered to be better, because it is based on a special case of "the ideal formula" put forth by

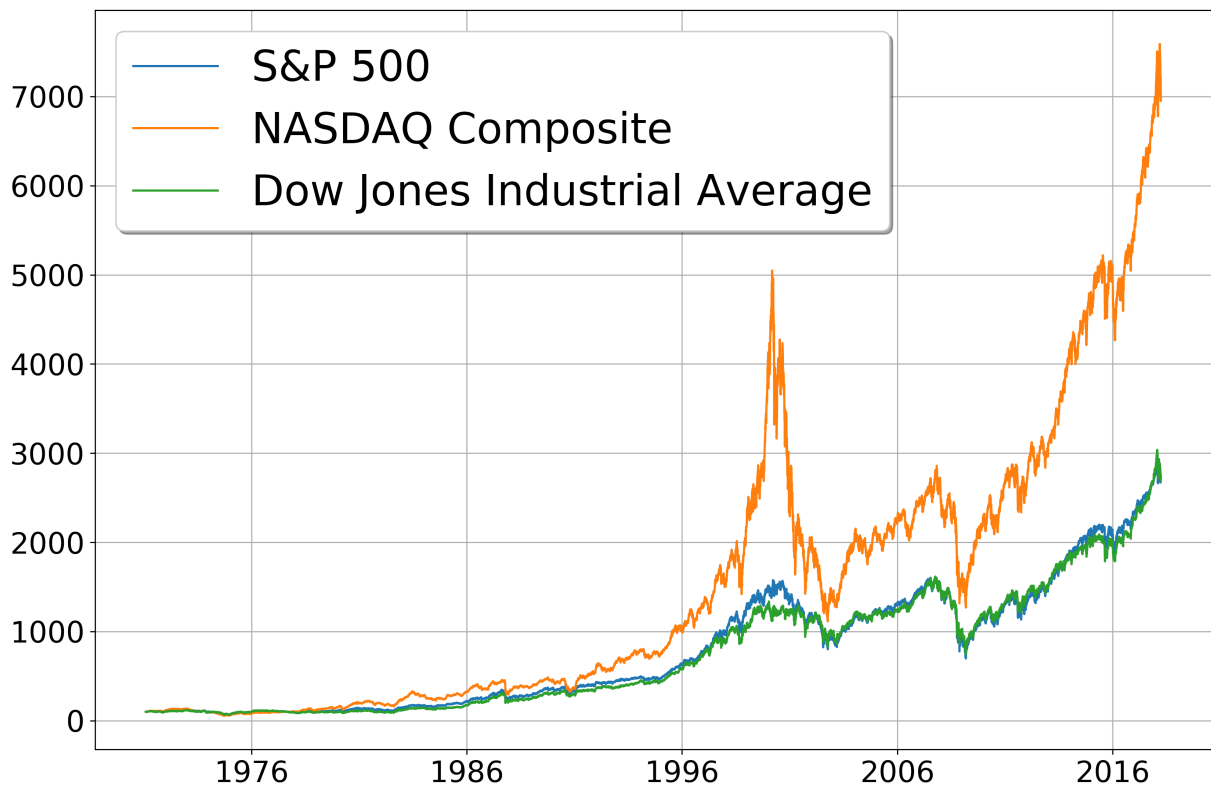
¹ Before venturing out on their own, they were working for the leading financial publisher of the day, Kiernan News Agency. While at Kiernan, Charles Bergstresser, known for his photographic memory, developed a stylus that could record news onto 35 sheets or bulletins simultaneously, a technique that quadrupled productivity and gave their fledging company a technological edge.

Fisher (1922). Computationally, it is more challenging since the market capitalization is the number of outstanding shares times the last traded price. Over the years, the S&P 500 index gained prominence and became a bellwether and leading indicator of the US economy.

After a few initial attempts, the first exchange traded fund (ETF) by the name of SPDR S&P 500 ETF Trust started trading about 100 years later. As the name suggests, this ETF is based on the S&P 500 index, which currently is regarded as the de facto proxy for the market portfolio by both the academia and the industry for the US market.

In 1971, NASDAQ began its all-electronic trading market. With this innovation, NASDAQ attracted new growth companies, such as Microsoft and Apple. On February 2, 1971, NASDAQ introduced its stock market index called the NASDAQ Composite Index at the inception value of 100 with 50 companies listed on its exchange. The calculation method uses the market capitalization method pioneered by S&P. Along with the Dow Jones Industrial Average and S&P 500, it is one of the three most-followed indexes in the US stock markets.

Figure 2.1 Comparison of 3 US Stock Market Indexes.



In Figure 2.1, we have re-based the Dow Jones Industrial Average and S&P 500 indexes to 100 on the same day of February 2, 1971. Clearly, over about 40 years, NASDAQ Composite index outperformed the other two established indexes, which are not too far apart. Also intuitively obvious is the fact that NASDAQ Composite index is the most volatile. What the chart shows is that if you had invested \$100 in 1971, the value of your investment at the end of about 40 years. The more volatile NASDAQ composite index would grow to about \$7,000 whereas the less volatile S&P 500 and DJIA indexes would yield about \$3,000.

Now, the value proposition in providing and licensing the indexes as intellectual properties saw the emergence of Morgan Stanley Capital International (MSCI) in 1969, when it began licensing its first equity index products. In the face of growing competition, interestingly, Dow Jones and S&P buried their hatchets and their merged to become S&P Dow Jones Indices. Another index provider is FTSE Russel. Together, these three index providers founded the Index Industry Association in March 2012 as an independent, not-for-profit organization. Its current members include niche index providers such as Hang Seng Indexes and Japan Exchange Group.

Today, we have a myriad of indexes, covering not just equity but also other asset classes: bond, currency, commodity, real estate, and so on. Some indexes are multi asset classes and some include derivatives such as futures and options, even credit default swaps.

On top of that, alternative methods to construct indexes have appeared. Indeed, index construction has taken on the color of portfolio management that is traditional the territory of fund managers. Notably, [Arnott, Hsu & Moore \(2005\)](#) propose the idea of fundamental indexes and they provide empirical evidence to suggest that he resulting portfolios outperforme the S&P 500 by an average of 1.97 percentage points a year over the 43-year span tested. A new bandwagon of alternative, fundamental, and smart beta indexes gathers as more and more ETFs based on these next generation indexes appear as new born stars in the universe of investables.

2.2 Index Weighted by Price

In the era when computing power and access to market information were limited, the easiest way to construct an index was to compute the average price of the component stocks.

2.2.1 Four Dow Jones Average Indexes

Take the 15-component DJUA index as an example. The last prices of the 15 component stocks as of May 18, 2018 are tabulated in Table 2.1.

Symbol	Company Name	Last Price
AEP	American Electric Power Company, Inc.	65.22
AES	The AES Corporation	12.05
AWK	American Water Works Company, Inc.	80.34
CNP	CenterPoint Energy, Inc.	25.29
D	Dominion Energy, Inc.	63.68
DUK	Duke Energy Corporation	74.15
ED	Consolidated Edison, Inc.	73.94
EIX	Edison International	61.22
EXC	Exelon Corporation	39.35
FE	FirstEnergy Corp.	33.25
NEE	NextEra Energy, Inc.	156.42
NI	NiSource Inc.	24.36
PCG	PG&E Corporation	42.22
PEG	Public Service Enterprise Group Incorporated	49.56
SO	The Southern Company	43.62

Table 2.1 Dow Jones Utility Index as at end of May 18, 2018.

To compute the DJUA index, you first sum up the prices, which is 844.67 in this example. You then divide this sum by the divisor 1.2634134826603 to obtain the index level of 704.377476, which is usually expressed up to two decimal places, i.e., 668.56.

Definition 2.1 Divisor is a numerical device that gives an index provider some level of flexibility to construct, compute, re-balance, and re-constitute the index. It gives the index constructor the flexibility to maintain the index. ■

In particular, when changing index constituents, the index constructor may adjust the divisor so that the value of the index with the new constituents equals the value of the index prior to the changes. Namely, the index constructor adjusts the value of the divisor to circumvent changes in the index value that are unrelated to changes in the prices of its constituent securities.

At the inception of an index, divisor is typically set in such a way that the index has a nice initial value, such as 100, as in the case of NASDAQ Composite Index.

This method of calculating the DJUA index value applies also to the DJIA and DJTA indexes. More recently, a composite index consisting of all the stocks in these three average indexes was constructed. So altogether, there are 4 price-weighted indexes for the US stock market. Officially they are described as

- Dow Jones Industrial Average

The index is a 30-stock, price-weighted index that measures the performance of some of the largest US companies. The index provides suitable sector representation with the exception of the transportation industry group and utilities sector, which are covered by the Dow Jones Transportation Average and the Dow Jones Utility Average, respectively.

- Dow Jones Transportation Average

The index is a 20-stock, price-weighted index that measures the performance of some of the largest US companies within the transportation industry group.

- Dow Jones Utility Average

The index is a 15-stock, price-weighted index that measures the performance of some of the largest US companies within the utilities sector.

- Dow Jones Composite Average

The index is a price-weighted measure of 65 US companies that includes all components of the Dow Jones Industrial Average, Dow Jones Transportation Average, and Dow Jones Utility Average.

Definition 2.2 A price-weighted index of n component stocks is formally computed as, given all the last traded prices $P_{i,t}$ of the component stocks at a given time t ,

$$I_t = \frac{\sum_{i=1}^n P_{i,t}}{d_s}.$$

The divisor d_s is last updated or adjusted at time $s \leq t$. ■

2.2.2 Nikkei 225 Index

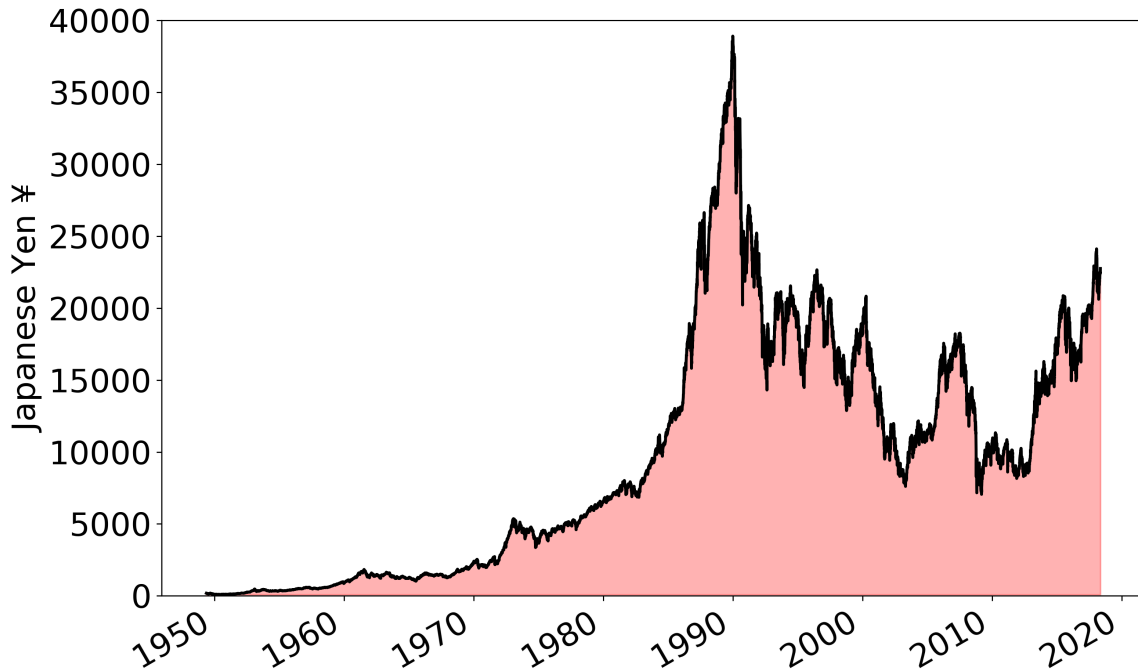
It turns out that most if not all of the stock market indexes in non-US countries is not price-weighted, except for Nikkei 225 index. In other words, only 4 Dow Jones Average indexes and Nikkei 225 index are price weighted.

Nikkei 225 index is calculated and published by a Japanese newspaper publisher—日本経済新聞 (Nihon Keizai Shimbun) or Nikkei in short. It consists of the 225 highly liquid stocks listed on the Tokyo Stock Exchange First Section. Since its inception on September 7, 1950, Nikkei 225 index has become an index widely followed as a barometer of the market or the state of the economy.

The historical Nikkei 225 index is plotted in Figure 2.2. We see that the index is calculated back to May 16, 1949 when the Tokyo Stock Exchange reopened after World War II. At that time, the Tokyo Stock Exchange calculated and announced the index as “TSE adjusted average price”. But when the Tokyo Stock Exchange started the TOPIX

index in 1970, Nikkei group took over and renamed it to “Nikkei Stock Average”, which is its formal name.

Figure 2.2 Nikkei 225 Index.



There is one slight variation in the calculation of Nikkei 225 index; the prices are to be adjusted before they are summed. The adjustment is based on the presumed par value. This is a historical vestige as in the past, every stock had a par value for the purpose of computing the dividend as a percentage of the par value, much like the coupon rate of a bond. The adjusted price of a constituent stock is simply the last trade price times 50 and then divided by the presumed par value of the stock, which by default is ¥50.

Definition 2.3 An adjusted-price-weighted index of n component stocks is formally computed as, given all the last traded prices $P_{i,t}$ and the presumed par values p_{i,s_i} of the component stocks at a given time t ,

$$I_t = \frac{\sum_{i=1}^n \frac{50}{p_{i,s_i}} P_{i,t}}{d_s}.$$

The divisor d_s is last updated or adjusted at time $s \leq t$. Also, $s_i \leq t$ for all $i = 1, 2, \dots, n$, which is the time when the presumed par value of stock i is last updated. ■

The reason for this adjustment is that stocks traded by the lot size of one share, which tend to have a higher par value (¥10,000) in the past and stocks traded by a lot size of 100 or 1000 shares with much smaller par values have different price levels. Intuitively, it is inappropriate to use the price of such different levels as they are to calculate the index. Nikkei circumvents this problem by using the “presumed” par value and adjusts the prices of the constituent to the presumed par value of ¥50 since the revision of the Commercial Law in October 2001. Being the major index for the financial products in the Japanese equity market Nikkei 225 is the underlying index for the popular financial products such as the index futures contracts and index funds.

It turns out that this variation from the standard price-weighted calculation has an unexpected benefit. If a stock undergoes a stock split, you just need to change its presumed par value. There is no need to change the divisor. In a stock split of x , the price will be adjusted on the effective day t :

$$P_{i,t} \longrightarrow xP_{i,t} =: \check{P}_{i,t}.$$

In the usual stock split when more shares are created, $x < 1$. On the hand, $x > 1$ in the case of a reverse split.

The split factor x can be absorbed by adjusting the presumed par value without changing the divisor:

$$p_{i,s_i} \longrightarrow xp_{i,s_i} =: p_{i,t_i},$$

and $t_i = t$. This is because

$$\frac{50}{p_{i,s_i}} P_{i,t} = \frac{50}{p_{i,s_i} x} P_{i,t} = \frac{50}{xp_{i,s_i}} xP_{i,t} = \frac{50}{p_{i,t_i}} \check{P}_{i,t}.$$

As an example, consider a [press release](#) dated March 16, 2018 regarding the reverse stock split of NH Foods, a constituent stock of Nikkei 225 index.

NH Foods Ltd. (2282), a Nikkei Stock Average (Nikkei 225) constituent, is planning a reverse stock split of 2 to 1. From the market open of the ex-right date, which is March 28, Nikkei Inc. will change the presumed par value of NH Foods from 50 yen to 100 yen.

Since the price level of NH Foods in the Nikkei 225 will stay the same, the divisor will not be changed by this event.

The factor x in this real-life 2-to-1 reverse stock split is 2. So in a way and in this way, the legacy of a par value is turned into a modern apparatus to manage stock splits.

2.2.3 How to Construct a Price-Weighted ETF?

Definition 2.2 indicates that all the prices are traded equally. To mimic a price-weighted index, you simply buy the same number or shares at the inception of an ETF.

As an example, suppose you buy 100 shares for each of the Dow Jones Utility Average (DJUA) index. Before transaction costs, from Table 2.1, the amount of money

needed is $\$889.92 \times 100 = \$8,899.20$. It is easy to see that the current value of your ETF is 100 times of

$$I_t d_s = \sum_{i=1}^n P_{i,t} = \$889.92.$$

In the case of Nikkei 225 index, because of the price adjustments before you can compute the index value, the equal-share approach does not work anymore. In principle, you buy each share according to the adjustment factor $\frac{50}{P_{i,s_i}}$. Put differently, the adjustment factor may be interpreted as the number of shares.

Since a stock cannot be traded in less than 1 share, you need to find a number N such that $N \times \frac{50}{P_{i,s_i}}$ is an integer for all $i = 1, 2, \dots, 225$. Given that the largest presumed par value is ¥500 as at May 11, 2018, the smallest possible $N = 500$. Thus, the number of shares you buy for Stock i is $\frac{25,000}{P_{i,s_i}}$ when you construct the ETF on Nikkei 225 index.

It is important to point out that the methods of creating an ETF described so far are void of regulatory and legal frameworks. Obviously, not anyone can create an ETF; you need to be an authorized participant (AP). An AP may be a market maker, prime broker-dealers, or any other large financial institutions. The method for Dow Jones Average indexes and the presumed par value adjusted method for the Nikkei 225 index described above are what an AP would use to acquire the shares needed for these equally-weighted ETFs.

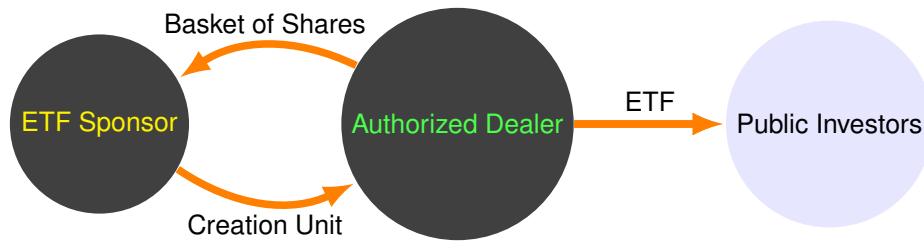
The process of an ETF creation begins when a prospective ETF trust manager (known as a sponsor) files an application with the regulator to obtain the license to create an ETF. Once the application is approved, the sponsor enters into an agreement with an AP. The AP acquires the necessary basket of shares in accordance to the weights of an index that the ETF is tacking, and delivers the basket to the ETF manager. In return, the AP receives the ETF shares in creation units. Every creation unit is usually a block of 50,000 ETF shares. Because this transaction is an in-kind transaction, i.e., securities are traded for securities, there are no tax implications, which is a major advantage of the ETF mechanism.

Once the authorized participant receives the creation units, they sell them to the public on the open market as ETF just like stock shares. They usually take on the role of a market maker for the ETF. The process of ETF creation is illustrated in Figure 2.3.

Meanwhile, the basket of securities that has been acquired to form the creation units remains in the trust account managed by the ETF trust manager. Generally, the trust manager has little activity beyond paying dividends from the stocks held in the trust to the ETF investors, and providing administrative oversight.

ETF redemption works in reverse, with the AP providing ETF shares to the ETF sponsor in return for underlying securities.

Figure 2.3 Illustration of ETF Creation.



After the ETF is listed on the exchange, the authorized participant may create more ETF shares if the demand from the general public is high. Conversely, redemption takes place if the ETF is not well received.

2.3 Index Weighted by Market Capitalization

Notice that in the price-weighted approach to constructing an index, the number of shares issued by each component stock is not needed. A disadvantage of this method, however, is that adjustments need to be performed, either on the divisor or the presumed par value, whenever a stock split occurs.

2.3.1 Value-Weighted Index

Consider instead market capitalization, which is basically a measure of the market value of a company. Stock split is a corporate action that involves neither the financial fundamental nor the value proposition of a company. Thus, intuitively, it makes sense to assert that the market capitalization remains invariant following a stock split.

Mathematically, it is easy to see why. First we define the market capitalization as follows:

Definition 2.4 Market capitalization of a company at time t , denoted by M_t , is defined as

$$M_t := N_s P_t.$$

Here P_t is the price of the stock per share at time t , and N_s is the number of outstanding shares issued by the company, correct as at time s , which of course is prior to time t . ■

It is easy to check that the market capitalization remains unchanged. Suppose t is the ex date of a stock split characterized by an adjustment factor f . As a result of the stock split, the number of shares becomes $S_t := fS_s$ and the share price becomes $\check{P}_t := P_t/f$. It turns out that

$$M_t = S_s P_t \frac{f}{f} = fS_s \times \frac{P_t}{f} = S_t \check{P}_t.$$

The convention is that in a stock split, $f > 1$, and in a reverse stock split, $f < 1$.

As information and computing technologies advance, it becomes easier to keep track of the number of outstanding shares and to calculate multiplication.

Definition 2.5 Let the market capitalization of a constituent stock be $M_{i,t}$. The index based on the market capitalization is then defined as

$$J_t := \frac{\sum_{i=1}^n M_{i,t}}{d_s} = \frac{\sum_{i=1}^n S_{s_i} P_{i,t}}{d_s}.$$

The resulting index is referred to as the **value-weighted index**. The divisor is last updated at time $s \leq t$, and S_{s_i} is the last updated number of shares for Stock i . ■

Though less often referred to, the value-weighted index can also be called size-weighted index because market capitalization is taken to be the size of a company in finance.

Proposition 2.6 If the divisor remains unchanged from time t to time u , then

$$J_u = J_t \times \frac{\sum_{i=1}^n S_{s_i} P_{i,u}}{\sum_{i=1}^n S_{s_i} P_{i,t}}. \quad (2.1)$$

Proof: By definition, at time t ,

$$d_s = \frac{\sum_{i=1}^n S_{s_i} P_{i,t}}{J_t}.$$

Likewise, at time u ,

$$d_s = \frac{\sum_{i=1}^n S_{s_i} P_{i,u}}{J_u}.$$

Equating these two expressions and after a simple algebraic move, the proposition is demonstrated to be true. ■

You may wonder why this apparently trivial proposition is included in the book. This is because historically, [Cowles 3rd \(1939\)](#) points out that the ratio on the right hand side of Equation 2.1 corresponds to a special case of the general ideal method

highlighted by [Fisher \(1922\)](#):

$$\sqrt{\frac{\sum_{i=1}^n S_{r_i} P_{i,u} \sum_{i=1}^n S_{s_i} P_{i,u}}{\sum_{i=1}^n S_{r_i} P_{i,t} \sum_{i=1}^n S_{s_i} P_{i,t}}}$$

In this expression, which [Fisher \(1922\)](#) reports was first proposed by Laspeyres in 1864, the time $r_i \leq s_i$, for all $i = 1, 2, \dots, n$, and $t < u$. A property highlighted by [Fisher \(1922\)](#) is that the expression is invariant with respect to

$$r_i \longleftrightarrow s_i.$$

This is a desired property in [Fisher \(1922\)](#). As a special case, if we let $S_{r_i} = S_{s_i}$, and we

obtain $\frac{\sum_{i=1}^n S_{s_i} P_{i,u}}{\sum_{i=1}^n S_{s_i} P_{i,t}}$ which is the ratio you see in Equation (2.1).

Using the same component stocks of Dow Jones Utility Average as of May 2018, in Table 2.3.1, we compare the contributions of every stock by two methodologies. One is the price-weighted methodology discussed earlier in Section 2.2. The contribution of Stock i is computed as $\frac{P_{i,t}}{\sum_{j=1}^n P_{j,t}}$. Note that the denominator is the sum of all prices.

The other is based on the market capitalization. Tabulated in Table 2.2 are the numbers of shares outstanding, with which the market capitalizations are computed according to Definition 2.4. The contribution w_i of Stock i relative to each other is thus given by

$$w_i := \frac{S_{s_i} P_{i,t}}{\sum_{j=1}^n S_{s_j} P_{j,t}}. \quad (2.2)$$

As you can see, AWK's contribution to DJUA is 9.51%. But its contribution to a would-be value-weighted index is only 3.34%. Conversely, Stock SO's contribution in DJUA is 5.16%, whereas it accounts for 10.18% of the total market capitalization from the 15 stocks with ticker symbols in Table 2.2.

It must be said that the number of outstanding shares does not remain constant. As can be seen from Figure 2.4, it can change more frequently than expected. The AES Corporation's numbers of outstanding shares plotted in Figure 2.4 as a time series are obtained from CRSP. The sample period is from June 25, 1991 through end of December 2018.

In particular, we have found 121 changes over a period of 26.5 years, i.e., at a rate of 4.57 changes per year. These changes are not caused by stock split or reverse stock

Ticker	Last Price	PW Weight	Shares Outstanding	Market Cap	MCW Weighted
AEP	65.22	7.72%	492,523,000	32,122,350,060	7.51%
AES	12.05	1.43%	661,400,000	7,969,870,000	1.86%
AWK	80.34	9.51%	178,048,000	14,304,376,320	3.34%
CNP	25.29	2.99%	431,473,000	10,911,952,170	2.55%
D	63.68	7.54%	652,552,000	41,554,511,360	9.71%
DUK	74.15	8.78%	701,000,000	51,979,150,000	12.15%
ED	73.94	8.75%	310,730,000	22,975,376,200	5.37%
EIX	61.22	7.25%	325,811,000	19,946,149,420	4.66%
EXC	39.35	4.66%	965,000,000	37,972,750,000	8.87%
FE	33.25	3.94%	476,909,000	15,857,224,250	3.71%
NEE	156.42	18.52%	471,000,000	73,673,820,000	17.22%
NI	24.36	2.88%	337,737,000	8,227,273,320	1.92%
PCG	42.22	5.00%	516,428,000	21,803,590,160	5.10%
PEG	49.56	5.87%	505,217,000	25,038,554,520	5.85%
SO	43.62	5.16%	999,000,000	43,576,380,000	10.18%

Table 2.2 Comparison of Contribution Weights by Price-Weighted (PW) and Market-Capitalization-Weighted (MCW) Methods, as at May 18, 2018. Open data source is [Nasdaq](#).

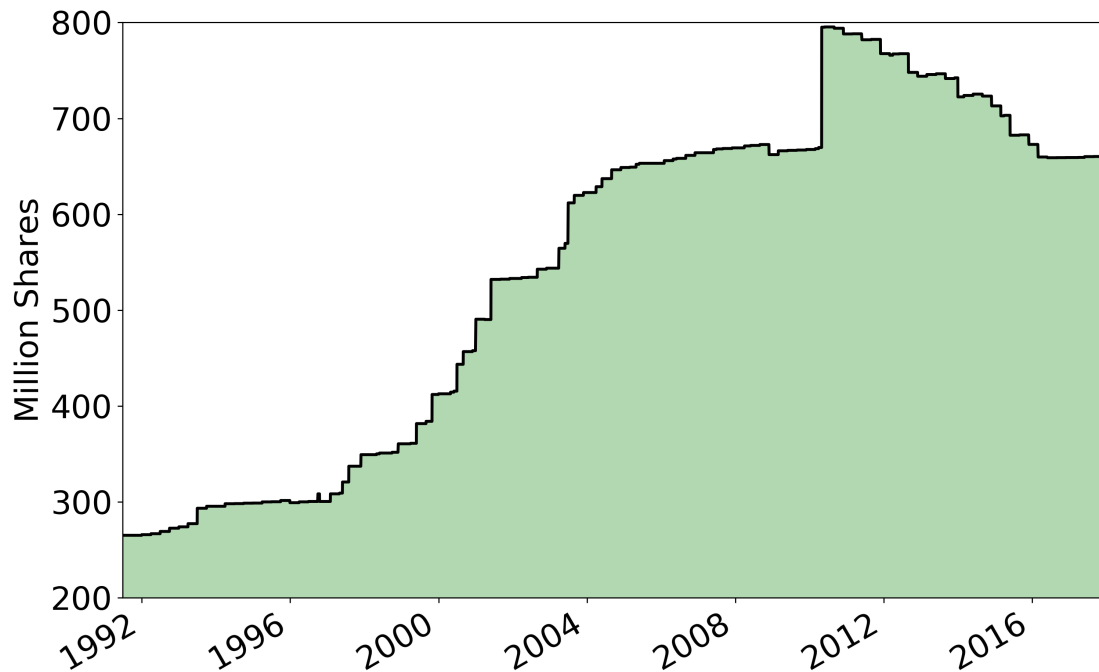
split because CRSP's data allow us to adjust for this corporate action. Overall, we see that the number of shares increases from about 265 million shares to about 800 million shares in May 2010. Since then, the company buys back the shares and reduces the number of outstanding shares to around 660 million shares by the end of December 2017. If the index maintenance policy is to adjust the divisor as long as and as soon as a component stock changes its number of outstanding shares, then the divisor of a value-weighted index—in principle—need to be updated very frequently, especially when the number of stocks is large, such as the S&P 500 index.

To reduce the frequency of update, S&P Dow Jones Indices sets a rule that says that changes in a company's total shares outstanding of 5% or more are treated on a weekly basis.

2.3.2 How to Construct a Value-Weighted ETF?

The method to construct a value-weighted ETF is different from the method of equal number of shares in Section 2.2.3 for creating an ETF after a price-weighted index.

First, we need to compute the weight according to Formula (2.2). The next thing to do is to apportion your funds according to the weights. Once a component stock is given the apportioned fund, you proceed to use the money to purchase the shares.

Figure 2.4 Number of Outstanding Shares of AES Corporation.

Obviously, you need to consider the costs of trading, something you don't have to think about when you construct an index. The algorithm to construct a value-weighted ETF is illustrated in the following pedagogical example.

EXAMPLE 2.1

You have \$100,000 dollars to invest in the VISE stocks², so as to create an ETF based on the value-weighted VISE index. The acronym VISE is coined for the following 4 companies, along with the exchanges where they are being listed and their respective ticker symbols.

- Visa (NYSE: V)
- Intuitive Surgical (NASDAQ: ISRG)
- Sirius XM Holdings (NASDAQ: SIRI)
- Electronic Arts (NASDAQ: EA)

² See the article by Sean Williams, [Forget "FANG" Stocks, and Say Hello to "VISE"](#).

The prices and number of shares outstanding of these companies, as well as the computations are tabulated in Table 2.3.

Ticker	V	ISRG	SIRI	EA
Name	Visa	Intuitive Surgical	Sirius XM Holdings	Electronic Arts
Last Traded	\$129.93	\$458.79	\$6.97	\$132.00
Shares Outstanding	1,786,164,000	112,299,000	4,491,864,000	306,728,000
Market Capitalization	\$232,076,288,520.00	\$51,521,658,210.00	\$31,308,292,080.00	\$40,488,096,000.00
Weight	65.30%	14.50%	8.81%	11.39%
Funds Allocated	\$65,301.07	\$14,497.04	\$8,809.45	\$11,392.44
Tentative Shares	502.59	31.60	1,263.91	86.31
Rounded Shares	503	31	1,264	86
Actual Funds Spent	\$65,354.79	\$14,222.49	\$8,810.08	\$11,352.00

Table 2.3 Construction of a Value-Weighted ETF with a basket of 4 stocks.

The total market capitalization, \$355,394,334,810, is the sum of those of the 4 VISE stocks. That is,

$$\begin{aligned} & \$232,076,288,520 + \$51,521,658,210 + \$31,308,292,080 + \$40,488,096,000.00 \\ & = \$355,394,334,810. \end{aligned}$$

To begin the VISE index at the level of 100, the initial divisor is set at

$$\frac{\$355,394,334,810}{100} = 3,553,943,348.10.$$

You can then proceed to compute the weight of each stock with respect to the total market capitalization.

Next, to construct an ETF based on the VISE index, you need to allocate your fund, say \$100,000, among the 4 VISE stocks. The weights you have computed earlier allow you to do so. You apportion the fund to each stock by multiplying \$100,000 with its weight. To obtain the tentative numbers of shares to buy, divide the funds allocated by the respective last traded share prices.

Finally, you need to exercise discretion to either round up or round down, because the smallest unit is one share. A key consideration here is that the actual funds spent should be very close to yet not exceeding \$100,000. As shown in the table, if you sum up the amounts needed to buy the rounded shares, you find that the actual total is

$$\$65,354.79 + \$14,222.49 + \$8,810.08 + \$11,352.00 = \$99,739.36,$$

which gives rise to a balance of \$260.64. Given the online commission of \$4.95 per trade, and the fact that the numbers of shares you need to buy are small, the left-over cash allows you to pay for the transaction costs. ■

2.3.3 Free Float

As discussed earlier, the rise of index-based mutual funds and ETFs is, to a large extent, due to the availability of equity indexes. These financial innovations in turn affect how the indexes are to be constructed. A major concern to the fund managers is the actual or effective number of shares that are publicly traded.

Suppose a company stock is a constituent of an index. The company has issued a total of 10,000 shares, out of which 5,100 shares are held by the company founders who intend to have an ultimate say on how the company is to be run. Another group of investors may have other strategic objectives. They buy and hold substantial percentage of the total number of shares, so as to, for example, sit on the board of the company. These shares held by company insiders are therefore not “floated” on the stock market for the general public to trade freely. Even though there are 10,000 shares, the actual number available for the general public to trade is only a fraction, say 25% of the total amount, or 2,500 shares in our illustration. If an ETF needs to acquire 2,600 shares of the company, then it becomes impossible since the supply is less than the required quantity. Moreover, the share price will rise drastically as soon as the market, by the animal spirit, detects a huge demand for these 2,500 shares.

Definition 2.7 Free float refers to the issued shares of a company that are in the hands of public investors, as opposed to shares closely held by investors who have agenda other than just an investment in the company. ■

Institutional investors prefer to invest in stocks with a large free float, as they can trade a significant number of shares without heavily impacting the share price. In 2001, the FTSE UK Index Series under FTSE Russell and MSCI pioneered the practice of making adjustments to the market capitalization³ by through the application of investability weightings to constituents’ share totals.

2.4 Case Study: Hang Seng Index

The Hang Seng 恒生 Index (HSI) is a market capitalization-weighted index of a selection of the largest and most liquid stocks listed on the Main Board of the Stock Exchange of Hong Kong. Like many major stock market indexes, free float adjustments

³ See [FTSE UK Index Series: History and heritage](#). MSCI in a press release announced that it would implement the adjustments of free float on or before June 30, 2001.

are applied. Furthermore, HSI has a 10% cap on the weight to avoid the domination by a single stock in the index.

Originally, the index was an in-house tool of Hang Seng Bank. It decided to launch Hang Seng Index on November 24, 1969 with 33 constituent stocks. HSI can be back-dated to July 31, 1964 with a starting base index level of 100. Since 2007, HSI has 50 stocks; it is the most widely quoted gauge of the Hong Kong stock market.

HSI is calculated and disseminated real-time every 2 seconds during trading hours on each trading day of the Hong Kong stock market. The formula to compute HSI consisting of 50 component stocks can be expressed as, with d_s being the most current divisor,

$$\text{HSI}_t = \frac{1}{d_s} \sum_{i=1}^{50} P_{i,t} S_{s_i} f_i c_i,$$

where

- $P_{i,t}$: last traded price of Stock i at time t
- S_{s_i} : most current number of issued shares of Stock i
- f_i : free float adjusted factor, which is between 0 and 1
- c_i : capping factor, which is between 0 and 1

EXAMPLE 2.2

At the March 2018 review, only two component stocks of HSI, HSBC and Tencent were given the capping factors of, respectively, 61.05% and 38.36%, so as to ameliorate their huge market capitalizations relative to the other 48 stocks, which do no need cap adjustment. Suppose the number of issued shares is obtained for each of the 50 constituent stocks. The adjusted market capitalizations (in Hong Kong \$) and weights or contributions to the index for HSBC and Tencent are computed as follows, as at end of May 21, 2018:

Name	Last Price	FFAF	CF	Issued Shares	FFA Shares	FFCA Market Cap	Weight
HSBC	\$77.8	100%	61.05%	20,378,431,083	20,378,431,083.0	\$967,912,303,306.14	10.07%
Tencent	\$408.0	60%	38.36%	9,503,193,536	5,701,916,121.6	\$892,400,049,892.27	9.29%

Apparently, the free float adjustment factor (FFAF) of Tencent is only 60%, and the free-float adjusted (FFA) number of shares is reduced by 40% to 5,701,916,121.6. For index, in contrast to ETF, having a decimal for the number of shares is not a problem since index is not directly tradable. The free-float and cap adjusted market capitalization is the product of the last price, the cap factor, and the FFA shares. ■

Definition 2.8 Investable Weight Factor (IWF) is the percentage of total shares outstanding that are included in the index calculation. ■

It may be more convenient to compute instead the investable weight factor as the product of the free-float adjustment factor and the capping factor:

$$IWF_i = FFAF_i \times CF_i$$

Thus, in the case of Tencent in Example 2.2, its IWF is $0.6 \times 0.3836 = 0.23016$, or 23.015%. In other words, only about 23% of the total number of issued shares are investible in the context of creating an ETF based on HSI.

2.5 Equally Weighted Index

By far, most of the representative stock market indexes are essentially value-weighted. This approach requires the index provider to update the number of issued or outstanding shares. A modern modification is that before the market capitalization of a component stock is computed, its free float percentage and other factors such as weight capping are taken into account to arrive at the effective or adjusted number of shares. Attempts to compute the value-weighted index are also compounded by the fact that the constituent companies may issue more shares, or buy back their shares. Moreover, most value-weighted indexes except a few of the MSCI licensed indexes do not disclose details concerning the adjusted number of shares.

In contrast, for the price-weighted approach explained in the earlier section, you are spared the tedium of accounting for the adjusted number of outstanding shares.

Another approach called the equally weighted or unweighted method also possesses this attractive feature of not being bogged down by the arduous tasks of looking up for the numbers of issued shares, free float factors, and so on.

Definition 2.9 Equally weighted index, also known as unweighted index, is defined as the index constituted by giving equal dollar amount and thus equal weight to every component stock. ■

Equally weighted index can be easily constructed by averaging the simple returns of the component stocks. At any given time, it is a cross-sectional average across all securities in the basket. If you chronologically collect the averages of the simple returns, you can construct an equally weighted index, as illustrated in the following example.

■ EXAMPLE 2.3

Suppose the averages of simple returns at Days 1 to 5 are, respectively, 1%, -2%, 3%, 10%, and -5%. You can start at Day 0 with a base index value of 100. Then on Day 1, the index value becomes $100 \times (1 + 0.01) = 101$. Likewise on Day 2, we have $101 \times (1 - 0.02) = 98.98$. On Day 3, the index value becomes 101.95, and 112.15 on Day 4. Finally, on Day 5, we obtain 106.54. ■

From the economic standpoint, equally weighted approach assigns equal weight to every constituent in the index, regardless of the company size, which is typically measured by market capitalization. Therefore, the smallest and largest companies receive the same weighting. Each constituent within an equally-weighted index exerts a similar impact on the overall performance. Concomitantly, equally-weighted indexes provide different sector exposures and risk-return profiles compared with the value-weighted counterparts.

2.5.1 Example: Value Line Index

As a practical example of an unweighted index, we have the Value Line Average index. It was introduced on June 30, 1961. This market benchmark assumes equally weighted positions in every stock covered in The Value Line Investment Survey, which is the flagship newsletter that tracks about 1,700 individual stocks.

The Value Line Average index assumes that an equal dollar amount is invested in each and every stock of the index. The returns from doing so are averaged geometrically every day across all these stocks in The Value Line Investment Survey. Consequently, this index is frequently referred to as the Value Line Geometric Average index. By covering a larger number of stocks than the S&P 500 index, and by giving equal weight to every stock, the Value Line Average index should give a better indication of the performance of the overall stock market as opposed to large-cap stocks or particular segments of the market.

On February 1, 1988, Value Line began publishing the Value Line Arithmetic Average index, in response to a need that subscribers and investors had. This new variant is also equally weighted. The difference lies in the mathematical technique employed to calculate the daily change. This arithmetic average is the one that we have dealt with in Example 2.3.

2.5.2 How to Create an Equally Weighted ETF?

The method to create an equally weighted ETF is quite straightforward. You simply apportion the total fund you have equally to every constituent stock.

As an illustration, consider the VISE value-weighted ETF discussed in Example 2.1.

 **EXAMPLE 2.4**

The fund of \$100,000 is divided equally among the 4 stocks. The rounded numbers of shares are tabulated as follows:

Ticker	V	ISRG	SIRI	EA
Name	Visa	Intuitive Surgical	Sirius XM Holdings	Electronic Arts
Last Traded	\$129.93	\$458.79	\$6.97	\$132.00
Funds Allocated	\$25,000.00	\$25,000.00	\$25,000.00	\$25,000.00
Tentative Shares	192.41	54.49	3,586.80	189.39
Rounded Shares	193	54	3,587	189
Actual Funds Spent	\$25,076.49	\$24,774.66	\$25,001.39	\$24,948.00

The total actual money spent is \$99,800.54, giving rise to a cash balance of \$199.46. This amount should be sufficient to pay for the transaction costs of acquiring the rounded shares.

Compared to the rounded shares in Example 2.1, Visa, which takes a lion share of 65.03% of the total fund, is drastically reduced to 25%. Consequently, the number of shares for Visa decreases from 503 shares to 193 shares.

2.5.3 Value-Weighted versus Equally-Weighted ETFs

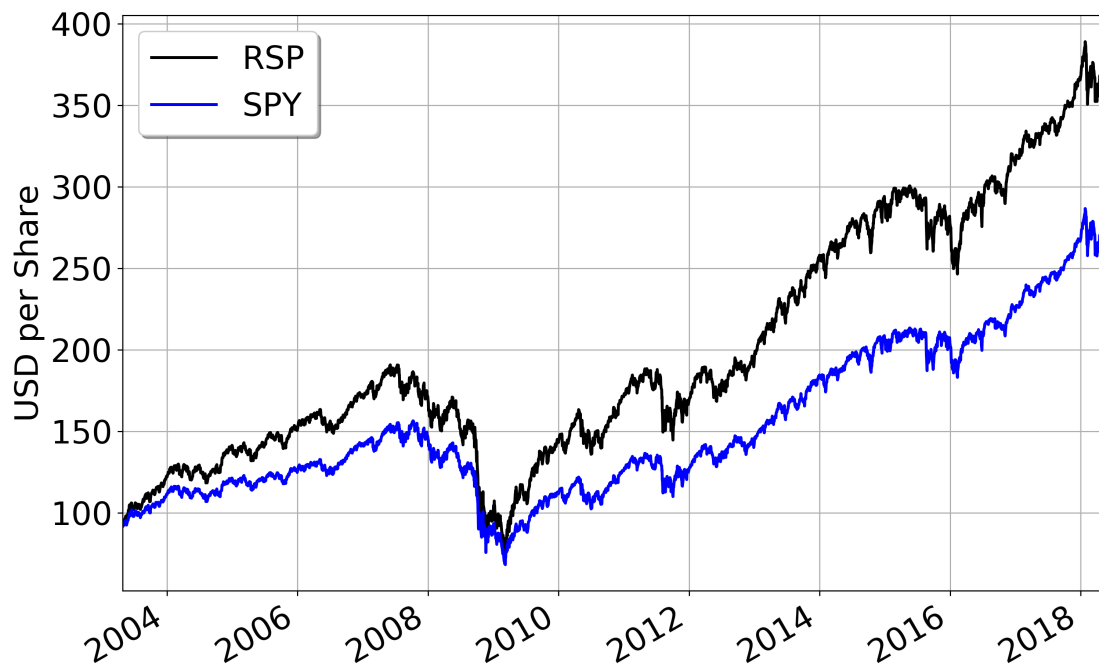
Under the ticker symbol SPY, the SPDR S&P 500 ETF seeks to provide investment results that, before expenses, correspond generally to the price and yield performance of the S&P 500 Index. The Trust seeks to achieve its investment objective by holding a portfolio of the common stocks that are included in the index, with the weight of each stock in the Portfolio substantially corresponding to the weight of such stock in the index. Launched in January 1993, SPY is the very first exchange traded fund listed in the United States. It has enjoyed enormous success since inception. As at end of March 2018, the market capitalization is about \$201.2 billion dollars, which is indicative of its popularity among investors.

On January 8, 2003, the S&P 500 Equal Weight index was launched, which could be back-dated to end of December, 1989. It is a size-neutral version of the S&P 500 index, i.e., having the same constituents as the value-weighted S&P 500 index. Each security issue in the S&P 500 Equal Weight index is allocated the same weight. About 3 months later, Invesco PowerShares listed the PowerShares S&P 500 Equal Weight ETF under the ticker symbol of RSP on the New York Stock Exchange's Arca platform. It is extolled to be the first smart beta ETF in the industry, and has the potential to outperform the traditional cap-weighted approach.

Naturally, investors would be interested to compare the performance of SPY versus RSP. For this purpose, we need to define the notion of net asset value (NAV) first.

Definition 2.10 NAV is the market value of a mutual fund's or ETF's total assets, minus liabilities, divided by the number of shares outstanding. The market value is determined by the midpoint between the bid-offer prices as of the closing time of the stock exchange on business days. ■

Figure 2.5 NAV Comparison of SPY and RSP



Historical NAV data for SPY and RSP are taken from Bloomberg. We scale RSP's NAV to SPY at the beginning for ease of visual comparison. Noticeable in Figure 2.5 is the clear out-performance of RSP over SPY.

What you can take away from Figure 2.5 is that if you invest a dollar on each of the two ETFs at the beginning of the sample period, i.e., the inception of RSP, you will get your reward from capital appreciation handsomely over about 15 years. Specifically, your every \$1 becomes about \$3 for SPY, and about \$4 for RSP.

What could possibly drive a wedge between these two ETFs? Despite having identical stocks as constituents, the different methods of weighing them are likely to cause the NAV to deviate from one another. We also observe from Figure 2.5 that when the market is bullish, the unweighted RSP tends to outperform the value-weighted SPY. Conversely, when the market is bearish, RSP seems to decline more than SPY.

2.6 Re-balancing

After an index has been created, the ongoing job of the index provider is to maintain it. There are many corporate actions and updates that will necessitate adjusting the divisor. Re-balancing refers to changing the divisor and other adjustment factors in response to a corporate action, after the trading hours. The basic principle underlying re-balancing is that the index level should not experience a jump as a result. In other words, the index level before and after re-balancing should remain unchanged.

2.6.1 Price-Weighted Index

An advantage of the price-weighted method is that it will continue to be price-weighted if there is no stock split. If the price of a stock increases more than other stocks in the index, its weight will increase. Conversely, the weight or contribution in the price-weighted index will decrease if the price decreases more than the rest.

Now, suppose a component stock experiences a stock split. As discussed in Section 2.3.1, everything else being equal, the price P_i will change to \check{P}_i . This price change will impact the price-weighted index and make it incorrect. Since stock split is a corporate action that will not change the profitability of a company, the price-weighted index should not change. Thanks to the divisor, we can adjust the divisor so that the index level remains invariant even when there is a stock split.

Without loss of generality, suppose Stock 1 has a stock split at time t . Using the pre-split price $P_{1,t}$, the index is computed as

$$I_t = \frac{1}{d_s} \left(P_{1,t} + \sum_{i=2}^n P_{i,t} \right).$$

To obtain the same I_t , we need to find a new divisor d_t so that

$$I_t = \frac{1}{d_t} \left(\check{P}_{1,t} + \sum_{i=2}^n P_{i,t} \right).$$

Solving for the new divisor d_t , we obtain

$$d_t = \frac{\check{P}_{1,t} + \sum_{i=2}^n P_{i,t}}{I_t} = d_s \times \frac{\check{P}_{1,t} + \sum_{i=2}^n P_{i,t}}{P_{1,t} + \sum_{i=2}^n P_{i,t}}. \quad (2.3)$$

Now, $\check{P}_{1,t} = P_{1,t}/f$ at the close of the market hours, where $f > 1$ for a stock split. The new divisor will be smaller than the old divisor d_s , as can be seen from Equation (2.3). Conversely, for reverse stock splits, $d_t > d_s$.

2.6.2 Value-Weighted Index

As discussed in Section 2.3.1, value-weighted index is invariant to stock splits. Hence, you don't need to adjust the value-weighted index when a stock split occurs.

The caveat, however, is that, whenever a company issues more shares, say more than 5% of the existing shares outstanding, the index provider must then adjust the divisor.

The principle of adjustment is still the same—the index must not change because of the additional shares. Without loss of generality, suppose the number of issued shares of Stock 1 increases by $\Delta S_{1,t}$, so that its market capitalization becomes

$$P_{1,t}S_{1,s_i} \longrightarrow P_{1,t}S_{1,s_i} + P_{1,t} \times \Delta S_{1,t}.$$

To annihilate the additional market capitalization, we need to adjust the divisor to

$$\begin{aligned} d_t &= \frac{P_{1,t}S_{1,s_i} + P_{1,t} \times \Delta S_{1,t} + \sum_{j=2}^n P_{j,t}S_{j,s_j}}{J_t} = d_s \times \frac{P_{1,t} \times \Delta S_{1,t} + \sum_{j=1}^n P_{j,t}S_{j,s_j}}{\sum_{j=1}^n P_{j,t}S_{j,s_j}} \\ &= d_s \left(1 + \frac{P_{1,t} \times \Delta S_{1,t}}{J_t} \right). \end{aligned} \quad (2.4)$$

From this equation, we see that the new divisor d_t will be larger than the old divisor d_s by a fraction $P_{1,t} \times \Delta S_{1,t} / J_t$, and vice versa.

Given that the corporate action events of issuance of more shares and share buy-backs occur more frequently than stock splits, it seems that updating of divisor for value-weighted index has to be carried out more often than for the price-weighted index.

2.6.3 Equally Weighted Index

Recall that an equally weighted index is one for which every stock has the same weight in the index, and a portfolio that tracks the index will invest an equal dollar amount in each applicable instrument. As stock prices move, the weights will shift and exact equality will be lost. Therefore, an equally weighted index must be rebalanced from time to time to re-establish equal weighting.

Same as any value-weighted index, the equally weighted index is also apathetic toward stock split. This is because when you buy an equal dollar amount of component stock, its market value is simply the market price times the number of shares hypothetically acquired at inception, i.e., $P_{j,t}S_{j,0} \times \text{IWF}_{j,0}$ for $j = 1, 2, \dots, n$. Here, recall that $\text{IWF}_{j,0}$ is the investible weight factor defined in Definition 2.8. As demonstrated earlier, the market value is not affected by stock split.

What about the issuance of new shares or shares buyback? Before answering this question, it is important to mention that index providers such as the S&P Dow Jones

Indices typically redefine the market capitalization for each stock used in the calculation of the equally weighted index. The principle of re-balancing, however, remains the same, i.e., after the market closes on day t ,

$$\text{Index level before re-balancing} = \text{Index level after re-balancing}$$

In addition to the investable weight factor IWF enunciated in Definition 2.8, a new adjustment factor is introduced in the market capitalization calculation to establish equal weighting. More concretely, an additional weight factor (AWF) is given to each stock. The following definition and the algorithm for re-balancing is based on the methodology of [S&P Dow Jones Indices \(2018\)](#).

Definition 2.11 Additional weight factor (AWF) is the adjustment factor of Stock i assigned at each index rebalancing date, t , which makes all index constituents' modified market capitalizations equal (and, therefore, equal weight), while maintaining the total market value of the overall index. ■

Now, let Z_0 be the hypothetical capital at the inception of an equally weighted index. We understand that Z/n amount is apportioned equally to each of the n stocks. Also, let s be the last re-balancing date, and t is today when re-balancing is to be performed. The algorithm for re-balancing goes as follows. First, we need to compute the market value of the index before re-balancing:

$$\sum_{j=1}^n P_{j,t} S_{s_j} \times \text{IWF}_{j,s} \times \text{AWF}_{j,s} = J_t d_s.$$

Note that only the index level J_t and the last-traded price $P_{j,t}$ are the most current; the other quantities were updated before time t .

Next, suppose the number of issued shares has changed from S_{s_j} to S_{t_j} , and the investible weight factor from $\text{IWF}_{j,s}$ to $\text{IWF}_{j,t}$. The new $\text{AWF}_{j,t}$ is re-calculated as

$$\text{AWF}_{j,t} = \frac{\frac{Z_0}{n}}{P_{j,t} S_{t_j} \times \text{IWF}_{j,t}}. \quad (2.5)$$

The denominator is the free-float adjusted market value of Stock j calculated from the updated data for the number of issued shares and its free float percentage. It can be easily recognized that at inception, $\text{AWF}_{j,0} = 1$.

In this way, the divisor after re-balancing is given by

$$d_t = \frac{\sum_{j=1}^n P_{j,t} S_{t_j} \times \text{IWF}_{j,t} \times \text{AWF}_{j,t}}{J_t}.$$

The numerator is none other than the index market value after re-balancing. Equivalently,

$$d_t = d_s \times \frac{\sum_{j=1}^n P_{j,t} S_{t_j} \times \text{IWF}_{j,s} \times \text{AWF}_{j,s}}{\sum_{j=1}^n P_{j,t} S_{s_j} \times \text{IWF}_{j,s} \times \text{AWF}_{j,s}}.$$

The upshot is that if a particular component stock issues a lot more shares, then we can just change the AWF of that stock using Equation (2.5), and keep the divisor as it is.

2.6.4 Section Summary

In this section, we deal with two corporate actions: stock splits and changes in issued shares. We summarize whether the divisor of an index must be adjusted or not in Table 2.4.

Method	Stock Split	Change in Issued Shares
Price-Weighted	Yes	No
Value-Weighted	No	Yes
Equally Weighted	No	No

Table 2.4 Summary of whether the divisor of an index must be changed during re-balancing.

Two remarks are in order. First, for the (modified) price-weighted Nikkei 225 index, because of the adoption of presumed par value, which is utilized to take care of stock split. As a result, divisor is unchanged when a component stock experiences a stock split. Second, for value-weighted indexes, to lower the frequency of re-balancing, the change in issued shares must be higher than a certain percentage, typically 5%. This is the rule index providers set as they go about maintaining the index.

2.7 Reconstitution

Every index has a set of eligibility criteria by which to decide whether a stock should or should not be included. Over time, a component stock may fail to satisfy all the criteria. The index provider will need to replace some of the existing component stocks that no longer are representative of the index anymore. Reconstitution is the process of deletion and addition of index's component stocks. It is parallel to portfolio managers changing the securities in their portfolios.

The overriding principle is that the index level is invariant before and after reconstitution.

2.7.1 Price-weighted index

The formula for reconstitution is exactly the same as re-balancing a stock split. In Equation (2.3), we simply interpret $\check{P}_{1,t}$ as the newly added security, and $P_{1,t}$ as the security deleted. Thus, a new divisor is obtained that ensures that the price-weighted index does not change.

EXAMPLE 2.5

Suppose there are 4 investible stocks, and their data are captured as follows:

Stock	A	B	C	D
Share Price	\$1	\$2	\$3	\$5

Suppose before reconstitution, a price-weighted index is based on Stock A, B, and C, and the divisor is 0.0075. The index level is therefore

$$(1 + 2 + 3)/0.0075 = 800.$$

Now, Suppose Stock A is to be replaced by Stock D. Using Equation (2.3), we obtain the new divisor

$$d_t = \frac{(2 + 3) + 5}{800} = 0.0125.$$

Note that we only need the information of share prices and other quantities are not required. ■

EXAMPLE 2.6

Consider an ETF based on the price-weighted index in Example 2.5. For pedagogical purpose, suppose the transaction cost is negligible. You, the ETF manager, have bought 3 million shares for each of the 3 stocks, as the asset under management of the ETF is 18 million dollars.

$$(\$1 + \$2 + \$3) \times 3,000,000 \text{ shares} = \$18,000,000.$$

In response to the reconstitution, how many shares must you have for each stock? The answer is easy to find as follows:

$$\frac{\$18,000,000}{\$2 + \$3 + \$5} = 1,800,000 \text{ shares}$$

Therefore, assuming zero trading cost, the transactions you must do are as follows:

1. Sell all 3 million shares of Stock A, and obtain the proceeds of 3 million dollars.
2. Sell $(3 - 1.8 =)$ 1.2 million shares of Stock B, and obtain the proceeds of 2.4 million dollars.
3. Sell 1.2 million shares of Stock C, and obtain the proceeds of 3.6 million dollars.
4. Use the total proceeds, i.e., $(3 + 2.4 + 3.6 =)$ 9 million dollars, to buy 1.8 million shares of Stock D at \$5 per share. Indeed,

$$\frac{\$9,000,000}{\$5} = 1,800,000 \text{ shares.}$$

As you can see, it is no mean feat to liquidate 3 million shares of Stock A. The market for Stock A will likely trend down as a result of this demand to sell. Likewise, for Stock B and Stock C, you also need to sell 1.2 million shares each. On the flip side, the market for Stock D will likely trend up because you must buy 1.8 million shares. These temporary imbalances are the results of reconstitution. ■

2.7.2 Value-Weighted Index

When one security is removed and another is added, the weights of *all* the other stocks in the basket must be changed. This is because the total market capitalization changes when an existing stock is replaced by another stock. As can be seen in Equation (2.2), since the total market capitalization is the denominator, the weight w_i of any stock i will change. Once the new weight for each stock is computed, you can then compute the number of shares to hold, and thereafter adjust the divisor so that the value-weighted index remains at the same level before reconstitution.

■ EXAMPLE 2.7

Consider again the same 4-stock market of Example 2.5, and information relevant to the value-weighted index is captured in Table 2.5. The adjusted number of shares N_i in Row 5 of Table 2.5 is computed as the product of the number of issued shares S_{s_i} , investible weight factor $IWF_{i,s}$, and additional weight factor $AWF_{i,s}$.

$$N_i = S_{s_i} \times IWF_{i,s} \times AWF_{i,s}.$$

Consequently, the adjusted market value denoted by $A_{i,t}$ of each stock is simply the share price times the adjusted number of shares.

$$A_{i,t} = P_{i,t} N_i.$$

Initially, we have Stocks A, B, and C as the constituents of the value-weighted index. The total adjusted market capitalization is the sum of adjusted market val-

Row		A	B	C	D
1	Share Price	\$1	\$2	\$3	\$5
2	Issued Shares	60,000,000	50,000,000	40,000,000	21,000,000
3	IWF	90%	50%	80%	100%
4	AWF	100%	100%	100%	99.0476191%
5	Adjusted Shares	54,000,000	25,000,000	32,000,000	20,800,000
6	Adjusted MV	\$54,000,000	\$50,000,000	\$96,000,000	\$104,000,000
7	Weight	27.0%	25.0%	48.0%	0.0%
8	New Weight	0.0%	20.0%	38.4%	41.6%

Table 2.5 Reconstitution of Value-Weighted Index.

ues (MV), which is calculated as

$$\$54,000,000 + \$50,000,000 + \$96,000,000 = \$200,000,000.$$

Now, suppose the divisor is 250,000 before reconstitution. The index level is

$$200,000,000 / 250,000 = 800.$$

The weight of each stock is found by dividing its adjusted market value by the total adjusted market capitalization.

Again, suppose Stock A is to be dropped and Stock D included. We need to compute the total adjusted market capitalization of the new combination, and the result is

$$\$50,000,000 + \$96,000,000 + \$104,000,000 = \$250,000,000.$$

To ensure that this value-weighted index remains at 800, the new divisor is computed as

$$d_t = \frac{250,000,000}{800} = 312,500.$$

Finally, the new weights based on the reconstituted total market capitalization are computed and reported in Row 8. ■

EXAMPLE 2.8

Suppose you are the manager of an ETF based on the value-weighted index of Example 2.7. Essential information about the stocks is the same as Table 2.5. Nevertheless, the share price, the adjusted market value (MV), and the weight of each

Row		A	B	C	D
1	Share Price	\$1	\$2	\$3	\$5
2	Adjusted MV	\$54,000,000	\$50,000,000	\$96,000,000	\$104,000,000
3	Weight	27.0%	25.0%	48.0%	0.0%
4	Fund MV	\$4,860,000	\$4,500,000	\$8,640,000	\$0
5	ETF Shares	4,860,000	2,250,000	2,880,000	0
6	New Weight	0.0%	20.0%	38.4%	41.6%
7	New ETF Fund Allocated	\$0	\$3,600,000	\$6,912,000	\$7,488,000
8	New ETF Shares	0	1,800,000	2,304,000	1,497,600

Table 2.6 Value-Weighted ETF and Response to Index Reconstitution.

stock is repeated in the first three rows of Table 2.6 for ease of reference.

Given that the asset under management is 18 million dollars, the fund of each stock moves in accordance to its weight in Row 3. As shown in Row 4, Stock C has the largest fund amount because its weight is 48%, which is the heaviest. The number of ETF shares for each stock is computed as the ETF divided by the stock price per share.

Again, suppose Stock A is to be dropped and Stock D is to be included. The fund allocations and ETF shares have to be changed in response to the new set of weights. These items are captured, respectively, in Row 7 and Row 8.

Assuming zero transaction cost, the concrete actions you must take are as follows:

1. Liquidate Stock A completely by selling 4.86 million shares to obtain 4.86 million dollars.
2. You need to sell $2,250,000 - 1,800,000 = 450,000$ shares of Stock B, since its weight is reduced from 25% to 20% as a result of reconstitution. You will obtain $\$2 \times 450,000 = \$900,000$.
3. Similarly, for Stock C, you need to sell $2,880,000 - 2,304,000 = 576,000$ shares. And you will have a cash flow of $\$3 \times 576,000 = \$1,728,000$.
4. The total cash flow from sales is

$$\$4,860,000 + \$900,000 + \$1,728,000 = \$7,488,000.$$

Use this money to buy Stock D. At the price of \$5 per share, the number of shares acquired is $\$7,488,000 / \$5 = 1,497,600$ shares.

■

2.7.3 Equally weighted index

Interestingly, the divisor is not changed when reconstitution is conducted for equally weighted index. This is because the company entering the index is given the adjusted market value of the company exiting the index. Consequently the total market value of the index neither increases nor decreases. Since the total market values does not change, there is no need to change the divisor.

Let the replacement Stock i_* 's share price be $P_{i_*,t}$; the number of issued shares be $S_{i_*}^*$; and the investable investment weight be $IWF_{i_*,t}$. These quantities are known or can be found for any given stock. The only unknown is the additional weight factor $AWF_{i_*,t}$. As the dollar amount of $A_{i,t}$ is assigned to Stock i , it follows that

$$AWF_{i_*,t} = \frac{A_{i,t}}{P_{i_*,t} S_{i_*}^* \times IWF_{i_*,t}}. \quad (2.6)$$

We have a one-for-one exchange of incoming and outgoing stocks in the reconstitution of an equally weighted index; other stocks in the basket are not affected. That said, if the reconstitution is not one-for-one—more added than deleted or vice versa—then all the component stocks will be affected.

EXAMPLE 2.9

Suppose we start with Stocks A, B, and C in Table 2.5 to constitute an equally weighted index. As expected, in between scheduled rebalancing, the market values of the component stocks are no longer equal. Specifically, Stock A's adjusted market value $A_{i,t}$ is 54 million dollars, as in Row 6 of Table 2.5. The adjusted market values of Stock B and Stock C are, respectively, 50 and 96 million dollars.

Suppose Stock A is to be replaced by Stock D. We simply assign the adjusted market value of Stock A to Stock D. Applying Equation (2.6), the new AWF for Stock D is

$$AWF_{D,t} = \frac{\$54,000,000}{5 \times 21,000,000 \times 100\%} = 51.42857142857143\%.$$

In this way, the total adjusted market value remains unchanged at \$200 million dollars. It follows that the index level is unchanged as well. Note that no adjustment is done on the divisor. ■

EXAMPLE 2.10

Suppose you manage an ETF based on the 3-stock equally weighted index. You apportion 18 million dollars equally among Stocks A, B, and C, each getting 6 million dollars to acquire these stocks. Over time, before the next scheduled re-

balancing, the adjusted market values change and they are reflected in Table 2.7.

Stock	A	B	C	D
Market Value	\$6,480,000	\$6,000,000	\$11,520,000	\$0
ETF Shares	6,000,000	3,000,000	2,000,000	0
New ETF Fund	\$0	\$6,000,000	\$11,520,000	\$6,480,000
New ETF Shares	0	3,000,000	2,000,000	1,296,000

Table 2.7 Impact of Reconstitution on Equally Weighted ETF.

In response to index reconstitution of Stock A being replaced by Stock D, you must liquidate all shares of Stock A. Assuming zero transaction cost, you then use the entire proceeds of \$6,480,000 to acquire 1,296,000 shares of Stock D at, say \$5 per shares:

$$\frac{6,480,000}{5} = 1,296,000 \text{ shares.}$$

Other stocks are not affected by the reconstitution. ■

2.8 Summary

Financial and economic indexes have progressively innovated into indispensable and multi-purpose devices. They are of great values for investors to gauge and track the investment performance, as well as to estimate risks. A security market index is a tool to measure the value of a set of securities in a target market, market segment, or asset class. The constituent securities selected for inclusion in the security market index are intended to represent the target market.

Substantially, indexes provide the basis for new investment products, including exchange-trade funds, mutual funds, other fund-based financial products, futures, and options. It is therefore important to understand the algorithms by which an index is constructed, re-balanced, and reconstituted. This knowledge will likely to provide a competitive edge if you are a money manager or a trader.

EXERCISES

2.1 Suppose you are employed as a junior quant in an ETF firm and you are given a task to construct a price-weighted index, a market-cap-weighted index, and an equally-weighted index. You have selected 3 stocks and their characteristics are as follows:

	Stock A	Stock B	Stock C
Initial price at Time 0	\$2.00	\$20.00	\$200.00
Shares outstanding	100,000,000,000	100,000,000	1,000,000
Initial market cap	\$200,000,000,000	\$2,000,000,000	\$200,000,000
Daily Simple Return at Time 1	-1.20%	0.80%	10.00%

- What is the level of a price-weighted index initially?
- What is the value of the divisor if you are to set the initial level of the market-cap weighted index at 100?
- Starting with a hypothetical value of \$14,400. How many shares of Stock B must you buy when you are constructing the equally-weighted index?
- What are the respective simple daily returns of the price-weighted, value-weighted, and unweighted indexes after the prices move to the new prices according to the respective daily simple returns?
- A new Stock D of price \$10 is to be added to the equally-weighted index at Time 1. What is the value of the new divisor?

2.2 Assume that the equity market has only 4 stocks. The number of free float (FF) shares is constant across Time 0 and Time 1 for each stock.

Stock		A	B	C	D
	FF Shares	20,000	25,000	50,000	40,000
Time 0	Price	\$35	\$68	\$72	\$100
	Volatility	80%	40%	60%	20%
Time 1	Price	\$40	\$75	\$79	\$110
	Volatility	72%	36%	54%	18%

You are an intern with a quantitative hedge fund from Time 0 to Time 1. Being a progressive hedge fund that believes in real-world investment, you are given \$40,000 to set up an ETF at Time 0. The instruction is that you must use up as close to \$40,000 as possible, i.e., as little left over cash as possible. You are to hold the ETF and re-balance it at Time 1. Trading cost is fixed at 0.5% of the trading amount in dollars. Suppose you want to construct an equally weighted ETF.

- Describe how you go about constructing the ETF at Time 0. (e.g., numbers of shares of A, B, C, and D; total cost of trading in constructing the ETF, etc)

- (b) Describe how you go about re-balancing the ETF so that it becomes equally weighted again. (e.g., numbers of shares of A, B, C, and D; total cost of trading in constructing the ETF, etc) You must once again make sure that as little cash is left over as possible.
- (c) As an intern wanting to secure a job, you have the drive to suggest a new idea for constructing a “smart beta” index at Time 1. Suppose the risk-free interest rate is 1%. You have an idea of using the Sharpe ratio as the criterion to give weight to each stock. What is the weight of each stock? Explain your answer.

Bibliography

Arnott, R. D., Hsu, J. & Moore, P. (2005), 'Fundamental indexation', *Financial Analysts Journal* **61**(2), 83–99.

Cowles 3rd, A. (1939), *Common Stock Indexes*, 2nd edn, Principia Press.

Fisher, I. (1922), *The Making of Index Numbers: A Study of their Varieties, Tests, and Reliability*, 2nd edn, Principia Press.

Lo, A. W. (2016), 'What is an index?', *Journal of Portfolio Management* **42**, 21–36.

S&P Dow Jones Indices (2018), 'Index mathematics methodology'.