QF 101 Revision

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UG Math

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Underlying Stochastic Process and Itô's Formula

Underlying Stochastic Process X_t

$$dX_t = \phi(X_t)\mu_t\,dt + \varphi(X_t)\sigma_t\,dB_t.$$

- Usually $\phi(x) = \varphi(x) = 1$. Then $\mu_t\,dt$ is the deterministic part and $\sigma_t dB_t$ is the random part.
- A very important special case is when $\phi(x) = \varphi(x) = x$.

  Question: What does this special case correspond to?

  Answer: ______________________

Simpler (1942) Version of Itô's Formula

Let $f(x)$ be a twice differentiable function: Then

$$df(X_t) = \frac{df}{dx}dX_t + \frac{1}{2}\frac{d^2 f}{dx^2}(dX_t)^2$$

Example 1

Suppose $\phi(x) = \varphi(x) = 1$, $\mu_t = \mu$ and $\sigma_t = \sigma$ are constants.

$$f(x) = \frac{x - \mu}{\sigma}$$

Step 1: $f'(x) = \dfrac{1}{\sigma}, f''(x) = 0$.

Step 2: Apply the simpler version of Itô's formula.

$$df(X_t) = \frac{1}{\sigma} dX_t + \frac{0}{2}\left(dX_t\right)^2$$

Step 3: Substitute in the underlying stochastic process.

$$\begin{aligned} df(X_t) &= \frac{1}{\sigma}\left(\mu dt + \sigma dB_t\right) \\ &= \frac{\mu}{\sigma} dt + dB_t \end{aligned}$$

Step 4: Integrate from time 0 to time t.

$$f(X_t) - f(0) = \frac{\mu}{\sigma} t + B_t.$$

Example 2

Suppose $\phi(x) = \varphi(x) = 1$, $\mu_t = \mu$ and $\sigma_t = \sigma$ are constants.

$$f(x) = \frac{a}{2}x^2 \quad \text{with } a \text{ being a constant.}$$

Step 1: $f'(x) = ax, f''(x) = a$.
Step 2: Apply the simpler version of Itô's formula.

$$df(X_t) = aX_t dX_t + \frac{a}{2}(dX_t)^2$$

Step 3: Substitute in the underlying stochastic process and apply Itô's table. for $(dX_t)^2$

$$df(X_t) = aX_t dX_t + \frac{a}{2}\sigma^2 dt$$

Step 4: Integrate from time 0 to time t.

$$\frac{a}{2}X_t^2 = a\int_0^t X_s dX_s + \frac{a}{2}\sigma^2 t \qquad \Longrightarrow \qquad \int_0^t X_t dX_t = \frac{1}{2}X_t^2 - \frac{1}{2}\sigma^2 t$$

Example 3

Suppose $\phi(x) = \varphi(x) = 1$, $\mu_t = \mu$ and $\sigma_t = \sigma$ are constants.

$$f(x) = \frac{a}{3}x^3 \quad \text{with } a \text{ being a constant.}$$

Step 1: $f'(x) = ax^2, f''(x) = 2ax$.
Step 2: Apply the simpler version of Itô's formula.

$$df(X_t) = aX_t^2 dX_t + aX_t (dX_t)^2$$

Step 3: Substitute in the underlying stochastic process and apply Itô's table. for $(dX_t)^2$

$$df(X_t) = aX_t^2 dX_t + aX_t \sigma^2 dt$$

Step 4: Integrate from time 0 to time t. (Suppose $X_0 = 0$)

$$\frac{a}{3}X_t^3 = a\int_0^t X_s^2 dX_s + a\sigma^2 \int_0^t X_s ds \Longrightarrow \int_0^t X_s^2 dX_s = \frac{1}{3}X_t^3 - \sigma^2 \int_0^t X_s ds$$

Example 4

Suppose $\phi(x) = \varphi(x) = X_t$, $X_t \neq 0$, $\mu_t = \mu$ and $\sigma_t = \sigma$ are constants. Hence, the SDE for X_t is

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t.$$

Step 1: Consider the function $f(x) = \log(x)$, so that $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$
Step 2: Apply the simpler version of Itô's formula.

$$\begin{aligned} d\log(X_t) &= \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} \left(dX_t\right)^2 = \mu \, dt + \sigma \, dB_t - \frac{\sigma^2}{2} dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \end{aligned}$$

Step 3 Integrate from time 0 to time t,

$$\log(X_t) - \log(X_0) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t \implies X_t = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t}$$

Financial Instruments and P&L

- Equity, Fixed Income, FX, Commodity
- Forward and Futures
- FRA, IRS, CIRS
- European and American Options

$$\text{P\&L (per unit) = selling price} - \text{buying price} = P_s - P_b \qquad (1)$$

Returns

- Gross return $R := \dfrac{P_s}{P_b}$
- Simple (rate of) return $r := \dfrac{P_s - P_b}{P_b} = R - 1$
- Log return $\ell := \log\big(1 + r\big) = \log P_s - \log P_b$
- Since asset prices cannot be negative, we have $-1 < r < \infty$.
- But $-\infty < \ell < \infty$.
- Because log function is concave, it must be that $\ell \leq r$.

Variance as a Difference of Two Returns

- At daily frequency or higher, the asset return r_t is generally very small in magnitude, i.e., $|r_t| < 1$.
- Pre-U's Maclaurin series suggests that

$$\boxed{\log(1 + r_t) = r_t - \frac{1}{2}r_t^2 + O(r_t^3). \qquad (2)}$$

- It follows that $r_t^2 \approx 2\big(r_t - \ell_t\big)$.
- Since the mean $\mathbb{E}\big(r_t\big) \approx 0$, $\mathbb{E}\big(r_t^2\big) \approx \mathbb{V}\big(r_t\big)$, i.e. the variance.
- Twice the difference between the simple return r_t and the log return $\ell_t := \log(1 + r_t)$ is the instantaneous variance.

$$\sigma_t^2 := 2\big(r_t - \ell_t\big).$$

FX Quoting Convention

- The bid and ask prices refer to the quoting currency, which is the currency after the "/" in the ISO convention:

Base Currency / Quoting Currency

- Customers buy at the higher ask price from the dealer and sell at the lower bid price to the dealer.
- Treat the currency as if it is a stock or gold.
- You can “short-sell” the base currency as easily as you take a long position.
- Unit or volume of transaction
  - x mil of base currency
  - y mil of quote currency

FX Market Practice

- The market practice has it that USD is always the base currency except
    - Euro: EUR
    - British Pound: GBP
    - Australian Dollar: AUD
    - New Zealand Dollar: NZD

- Interestingly, dealers trade these currencies by their nicknames: Fiber for EUR, Sterling for GBP, Aussie for AUD, and Kiwi for NZD. The U.S. dollar is nicknamed the Greenback or Buck, Swiss franc the Swissy, Canadian dollar the Loonie, and so on

Compounding Schemes

- By default, all interest rates are quoted on the annualized basis.
- Discrete compounding given the interest rate r_d

$$\text{FV} = \text{PV} \times \left(1 + \frac{r_d}{n}\right)^{nT}.$$

- Money market: When $T < 1$, the compoudning scheme is $1 + r_d T$.
- Two important frequencies
  - Semi-annual: $n = 2$
  - Monthly: $n = 12$
- Continuous compounding given the interest rate r_c

$$\text{FV} = \text{PV} \times e^{r_c T}$$

Effective Annual Rate

- The annual compounding rate is also known as the simple interest rate.
- To compare different compounding schemes, a common practice is to entertain the notion of effective annual rate $\widehat{r}$, which is the interest rate that would be obtained if the forward value were to be calculated under the annual compounding scheme.
- For example, the rate r of continuous compounding is equivalent to $\widehat{r}$ via the following equation:

$$\text{FV} = \text{PV} \times e^{rT} = \text{PV} \times \big(1 + \widehat{r}\big)^{T}.$$

In other words, $\widehat{r} = e^{r} - 1$.

Fixed Income

- Financial Industry Regulatory Authority's Market Data
- Quoting convention: Percent of par value expressed in \$
- One-to-one mapping of yield to maturity and price for a fixed-coupon bond of coupon rate c and T years to maturity:

$$\boxed{\begin{aligned} p &= \frac{c}{2}\sum_{k=1}^{2T}\frac{1}{\left(1+\dfrac{y}{2}\right)^{k}} + \frac{1}{\left(1+\dfrac{y}{2}\right)^{2T}} \qquad (3) \\ &= \frac{c}{2}\sum_{k=1}^{2T}\frac{1}{\left(1+\dfrac{z_k}{2}\right)^{k}} + \frac{1}{\left(1+\dfrac{z_{2T}}{2}\right)^{2T}} \qquad (4) \end{aligned}}$$

Spot zero Rates and Par Rates

- The term structure of Treasury's zero rates z_k is the yield curve.
- Price (present value) of a discount bond with face value of 1$,

$$\boxed{\mathsf{PV}_k = \frac{1}{\left(1+\frac{z_k}{2}\right)^k} = \mathsf{DF}_k, \qquad (5)}$$

  is also known as the discount factor. The tenor of this discount bond is k half-years.
- From spot zero rates, you can compute the par rate c_k given $p = 1$ by

$$1 = \frac{c_k}{2}\sum_{i=1}^{k}\frac{1}{\left(1+\frac{z_i}{2}\right)^i} + \frac{1}{\left(1+\frac{z_k}{2}\right)^k}.$$

- In this way, you can obtain a term structure of par rates.

Class Exercise: Sample Question 1

- Suppose the term structure of (fictitious) zero rates is given below:

k half-years	1	2	3	4
z_k	0.7%	1.2%	1.8%	2.0%

- Compute the term structure of par rates c_1, c_2, c_3 and c_4.

Forward Interest Rates

- The spot rate is essentially the geometric average of the forward-forward rates.

$$\boxed{\left(1+\frac{z_k}{2}\right)^k = \left(1+\frac{f_{(0,1)}}{2}\right)\left(1+\frac{f_{(1,2)}}{2}\right)\cdots\left(1+\frac{f_{(k-1,k)}}{2}\right) \quad (6)}$$

- The implicit relationship between the spot and forward interest rates is

$$1+\frac{f_{(k-1,k)}}{2} = \frac{\left(1+\frac{z_k}{2}\right)^k}{\left(1+\frac{z_{k-1}}{2}\right)^{k-1}} = \frac{\mathsf{DF}_{k-1}}{\mathsf{DF}_k}.$$

Class Exercise: Sample Question 2

- Given the spot rates in Question 1 (Slide 18), construct the correponding discount factors.

- Based on the discount factors, construct the term structure of forward rates.

Principles of $\mathbb{Q}$uantitative Finance

- According to the first principle, the Treasury zero rates should be all equal, i.e., the yield curve should be flat, if there is absolutely no risk.
- Based on the second principle, the long-term and short-term risks render the term structure into a curve with level, slope, and curvature.
- A parsimonious model of 1- to 10-year yield curve is

$$Y_T = r + \beta^{(l)} T - \beta^{(s)} \left(\frac{1 - e^{-T/\tau}}{\frac{T}{\tau}} - e^{-T/\tau} \right).$$

- The third principle provides the mechanism by which the first and second principles are observed in the market.

Interest Rate Risk and Return

- The change in interest rate δr is, in percentage terms,

$$\boxed{\triangle r := \frac{\delta r}{1+r} \qquad (7)}$$

- In terms of $\triangle r$, the return corresponding to the interest rate risk is

$$\boxed{R_s = -D\triangle r + \frac{1}{2}C(\triangle r)^2, \qquad (8)}$$

where the duration D and convexity C are, respectively,

$$\boxed{D := (1+r)D_m, \qquad C := (1+r)^2 C_m. \qquad (9)}$$

- The modified duration D_m and modified convexity C_m are, respectively,

$$D_m := \frac{1}{P}\frac{\partial P}{\partial r}, \qquad C_m := \frac{1}{P}\frac{\partial^2 P}{\partial r^2}.$$

Linear Payoff

Forward Price F_0

- Time to maturity T, asset's spot price S_0, and risk-free rate r_0
- Forward price

$$F_0 = S_0(1 + r_0 T) \tag{10}$$

- Payoff at maturity for the buyer: $S_T - F_0$

Interest Rate Parity and Forward FX f_0

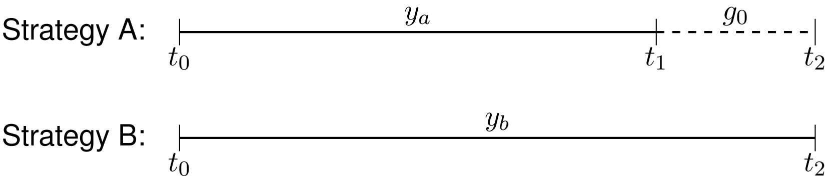
- Risk-free rate for quote currency $r_q = r_0$, risk-free rate for base currency r_b,

$$f_0 = \frac{S_0\big(1 + r_0 T\big)}{1 + r_b T}. \tag{11}$$

- Payoff at maturity for the buyer: $S_T - f_0$

Forward Interest Rate

- y_a: risk-free yield of tenor $t_1 - t_0$
- y_b: risk-free yield of tenor $t_2 - t_0$
- g_0: (implied) forward interest rate



Two Strategies that Give Rise to the Same Forward Value

Forward Interest Rate (Cont'd)

- By the first and third principles of $\mathbb{Q}$F,

$$(1 + y_a)^{t_1 - t_0} \times (1 + f_0)^{t_2 - t_1} = (1 + y_b)^{t_2 - t_0} \tag{12}$$

- Solving for f_0, we obtain

$$f_0 = \left(\frac{(1 + y_b)^{T_2}}{(1 + y_a)^{T_1}} \right)^{\frac{1}{T_2 - T_1}} - 1.$$

- For notational convenience, we have let $T_1 := t_1 - t_0$ and $T_2 := t_2 - t_0$.

FRAs of Short-Term Maturities

✳ The fair value K is given by the following relationship:

$$(1+\tau_1 r_1)(1+\tau_k K) = 1 + (\tau_1+\tau_k) r_2, \tag{13}$$

where

- r_1 is the spot rate with a shorter maturity τ_1.
- τ_k is the FRA maturity
- r_2 is the spot rate with maturity $\tau_1+\tau_k$.

✳ It follows from (13) that the FRA rate is given by

$$K = \frac{1}{\tau_k}\left(\frac{1+(\tau_1+\tau_k)r_2}{1+\tau_1 r_1} - 1\right). \tag{14}$$

Payoff of FRA is Linear

- At time τ_1 when the FRA expires, the LIBOR rate R of tenor τ_k is observed. The cash flow to the buyer is then given by

$$\text{Notional Amount} \times (R-K)\tau_k \left(\frac{1}{1+R\tau_k} \right).$$

- The cash flow generated by the interest rate differential is discounted by the discount factor $\dfrac{1}{1+R\tau_k}$.
- This is because instead of entering into the “physical” or actual borrowing over the tenor of τ_k starting from τ_1, the anticipated cash flow at $\tau_1 + \tau_k$, namely, notional Amount $\times (R-K)\tau_k$, is settled at τ_1 by discounting it back from $\tau_1 + \tau_k$ to τ_1.

Pricing of IRS' Swap Rate K

- The net present value of the IRS at time 0 is

$$\begin{aligned}\text{NPV}_0 = {} & \left(\sum_{j=1}^{n} \text{DF}_j \times \text{Floating CF}_j + \text{DF}_n \times 1 \right) \\ & - \left(\sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i + \text{DF}_n \times 1 \right).\end{aligned}$$

- In this form, IRS is effectively a long-short strategy on two bonds. The IRS buyer is effectively betting on a position that is long in the floating rate security and short in the fixed rate bond.

Pricing of IRS' Swap Rate K (Cont'd)

- At time 0, since both bonds are issued at par, by the third law of $\mathbb{Q}$F, we must have NPV$_0 = 0$. Accordingly, we set the floating bond to its par value to obtain

$$0 = 1 - \sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i - \text{DF}_n \times 1.$$

- Result: Pricing the IRS' swap rate K per period (e.g. semi-annual)

$$\boxed{K = \frac{1 - \text{DF}_n}{\sum\limits_{i=1}^{n} \text{DF}_i}. \qquad (15)}$$

Overnight Index Swaps (OIS)

- Overnight indexed swaps are interest rate swaps in which a fixed rate of interest (OIS rate) is exchanged for a floating rate that is the geometric mean of a daily overnight rate.
- The overnight rates include
  - Federal Funds rate (USD)
  - EONIA (EUR)
  - SONIA (GBP)
  - CHOIS (CHF)
  - TONAR (JPY)
- There has recently been a shift away from LIBOR-based swaps to OIS indexed swaps due to the scandal.
- Discounting with OIS is now the standard practice for pricing collateralized deals and is being mandated by clearing houses.

NPV Pricing of CIRS' Swap Rate K

- Given the spot FX rate S_0, which is the units of quote currency needed to exchange for one unit of base current, the net present value for the CIRS buyer is

$$\begin{aligned}\text{NPV}_0 =& S_0 \left(\sum_{j=1}^{n} \text{DF}_j \times \text{Floating CF}_j + \text{DF}_n \times 1 \right) \\ & - \left(\sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i + \text{DF}_n \times 1 \right).\end{aligned}$$

- The buyer receives the base currency in exchange for the quote currency at the spot rate S_0.

NPV Pricing of CIRS' Swap Rate K (Cont'd)

- Again, this is a long-short strategy. The CIRS buyer is long a floating bond denominated in the base currency and short in a fixed rate bond in the quote currency.
- What is the value of NPV_0 at time 0?

  Answer: ________________________________
- Floating leg's bond is valued at par.

$$S_0 - 1 = S_0 - \left(\sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i + \text{DF}_n \times 1 \right).$$

- Solving for K, we find that the fixed rate is still given by the same formula: (15)!

Options

- Contract specification: call or put, strike price K, maturity T, exercise style
- Underlying “asset” S_t, risk-free interest rate r_t
- Option pricing depends on volatility of the underlying, a lot!
- Money-ness
- Intrinsic value, time value, early exercise premium (for American option)

Put-Call Parity

- The net cash flow at time T is zero, regardless of the outcomes (either $S_T < K$ or $S_T > K$ or $S_T = K$).
- By the first principle of $\mathbb{Q}$F, the cash flow at time 0 must also be zero because there is no uncertainty and hence no risk. Why no uncertainty? All the prices and the interest rate are known at time 0!
- Hence

  $$Ke^{-rT} + c_0 - S_0 - p_0 = 0.$$

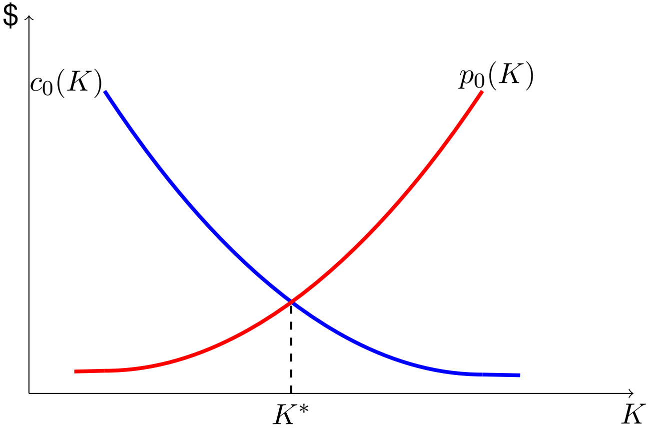
  and this put-call parity is more commonly written as

  $$c_0 - p_0 = S_0 - Ke^{-r_0 T}.$$

- At time t, it is written as

$$\boxed{c_t - p_t = S_t - Ke^{-r_t(T-t)} \qquad (16)}$$

Option Price Curves as Functions of Strike K



Monotonicity, Gradient Boundedness, and Convexity

- $K_1 < K_2 < K_3$
- Monotonicity in the option price level

$$\boxed{c_0(K_2) \leq c_0(K_1); \qquad p_0(K_1) \leq p_0(K_2).} \tag{17}$$

- Boundedness in the gradient

$$\boxed{-1 \leq \frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq 0; \qquad 0 \leq \frac{p_0(K_2) - p_0(K_1)}{K_2 - K_1} \leq 1.} \tag{18}$$

- Convexity

$$\boxed{\frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq \frac{c_0(K_3) - c_0(K_2)}{K_3 - K_2}; \quad \frac{p_0(K_2) - p_0(K_1)}{K_2 - K_1} \leq \frac{p_0(K_3) - p_0(K_2)}{K_3 - K_2}.} \tag{19}$$

Static Replication

$$\begin{aligned} f(S) &= f(\lambda) + f'(\lambda)(S-\lambda) \\ &\quad + \int_0^{\lambda} f''(K)(K-S)^+ dK + \int_{\lambda}^{\infty} f''(K)(S-K)^+ dK \end{aligned} \tag{20}$$

- The payoff $f(S)$ contingent on the outcome S at maturity T can be replicated by
  - $f(\lambda)$: number of risk-free discount bonds, each paying \$1 at T
  - $f'(\lambda)$: number of forward contracts with delivery price λ
  - $(K-S)^+$: European put option's payoff at T of strike K
  - $(S-K)^+$: European call option's payoff at T of strike K
  - $f''(\lambda)dK$ is the number of put options of all strikes $K < \lambda$, and call options of all strikes $K > \lambda$
- The payoff replication is static, and model-free of Type 1.

Model-Free Approach to VIX

$$\sigma_{\text{MF}}^2 T = 2e^{r_0 T}\left(\int_{F_0}^{\infty} \frac{c_0}{K^2}\,dK + \int_0^{F_0} \frac{p_0}{K^2}\,dK\right). \tag{21}$$

- No requirement for an option pricing model

  ⟹ No model risk!
- No worry about parameters
  - The only exogenous inputs are risk-free interest rate and dividend yields
- No bias
  - σ_{MF} reflects volatility across all out-of-the-money strike prices and thus reflects the option skew
- Uses both put and call options

  ⟹ σ_{MF} is less sensitive to individual option prices.
- The formula is beautiful!

Binomial Tree Model for Option Pricing

- In addition to the stock price S_0, the most important quantity needed for option is volatility σ.
- A model for up and down factors is

$$u = e^{\sigma\sqrt{t}}, \qquad \text{and} \qquad d = e^{-\sigma\sqrt{t}}.$$

- For each t of the binomial tree, the risk-neutral valuation of a pair of future payoffs is

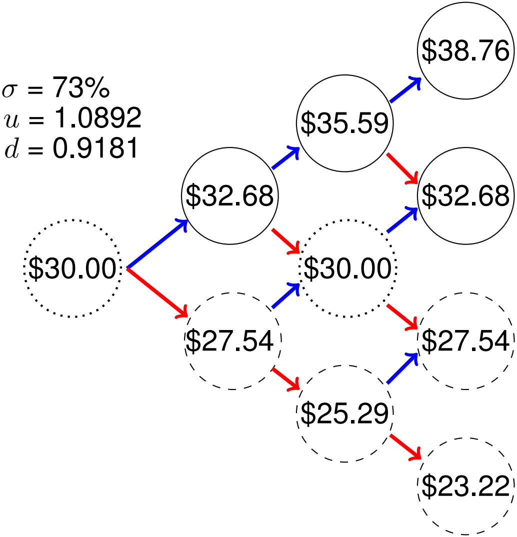
$$c_t = e^{-r_0}\left(pc^+_{t+1} + (1-p)c^-_{t+1}\right) = e^{-r_0}\mathbb{E}\left(c_{t+1}\right), \tag{22}$$

where the risk-neutral probability of up movement is

$$\boxed{p = \frac{e^{r_0} - d}{u - d}. \tag{23}}$$

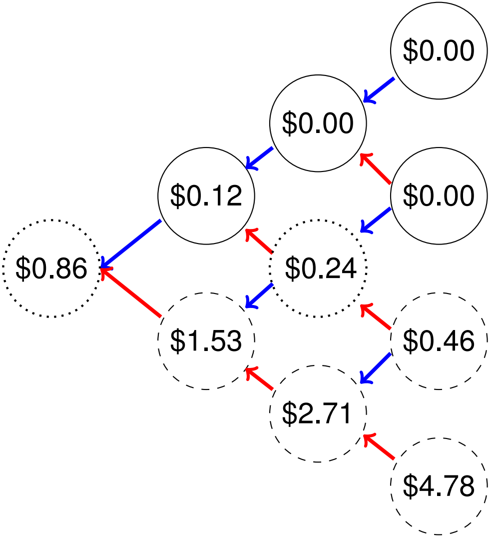
A Numerical Example of Binomial Option Pricing

- Asset prices for all nodes
- S_0 = ________
- Put option's days to maturity = 15 days
- Since $N = 3$, each period is 15/3 = 5 days
- 5 days is t = 5/365 = 1/73 years
- risk-free rate r_0 = 0.25%



Put Option Prices

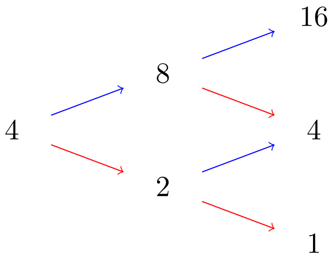
- Strike price = $28
- Upward probability p = 47.89%



Two-Period Binomial Tree Algorithm

✳ Two-step binomial tree given by the parameters:

- $S_0 = 4$
- $u = 2$
- $d = 1/2$
- $r = 22.31\%$ (artificially made very large to get nice numbers)
- $\Delta t = 1$

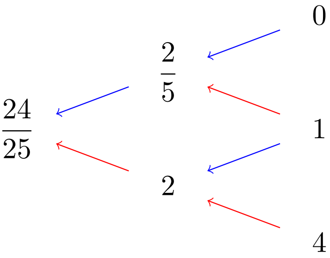


Two-Period Binomial Tree for European Put

- Compute the risk-neutral probability of upward movement p, and set $q := 1 - p$.
- To value a European put option struck at $K = 5$, we evaluate

$$V_n = e^{-r\Delta t}\mathbb{E}_n^{\mathbb{Q}}\left(V_{n+1}\right) = e^{-r\Delta t}\left(pV_{n+1}^{+} + qV_{n+1}^{-}\right).$$

- The result is $V_0 = p_0 = \dfrac{24}{25}$

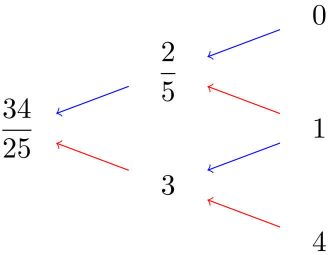


Binomial Tree for American Put

- At each time step prior to the expiry nodes, the early exercise provision in the American option gives you the choice of either to exercise immediately and receive the intrinsic value of the option, or to hold on to the option to the next step.

$$V_n = \max\left(e^{-r\Delta t}\left[pV_{n+1}^{+} + qV_{n+1}^{-}\right], (K - S_n)^{+}\right).$$

- Continuing from the earlier example in Slide 42,



$\frac{34}{25}$

$\frac{2}{5}$

3

0

1

4

Model-Free Properties of American Options

- Put-call Inequality

$$\boxed{S_0 - K \le C_0 - P_0 \le S_0 - Ke^{-rT}. \qquad (24)}$$

- Irrational to early exercise American calls on stocks that don't pay dividends

$$C_t \ge c_t > S_t - K$$

- Irrational to early exercise American puts on stocks that don't pay dividends *and* when $C_t \ge K\left(1 - e^{-r(T-t)}\right)$
- Irrational to early exercise a margined put or call option on futures

Binomial to Continuous

- Binomial random walk becomes a Brownian motion as $\Delta t \longrightarrow 0$.
- Einstein and Bachelier's theories lead to a proportional relationship between variance and time.
- Binomial tree pricing model becomes the Black-Scholes pricing formula as the number of periods becomes very large.
- In the original Blakc and Scholes (1973), Itô's calculus is needed to arrive at the Black-Scholes equation.
- The Black-Scholes model works for European options only

Black-Scholes Option Pricing Formulas

- d_1 and d_2

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},$$

$$d_2 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},$$

- Standard normal cumulative distribution function:

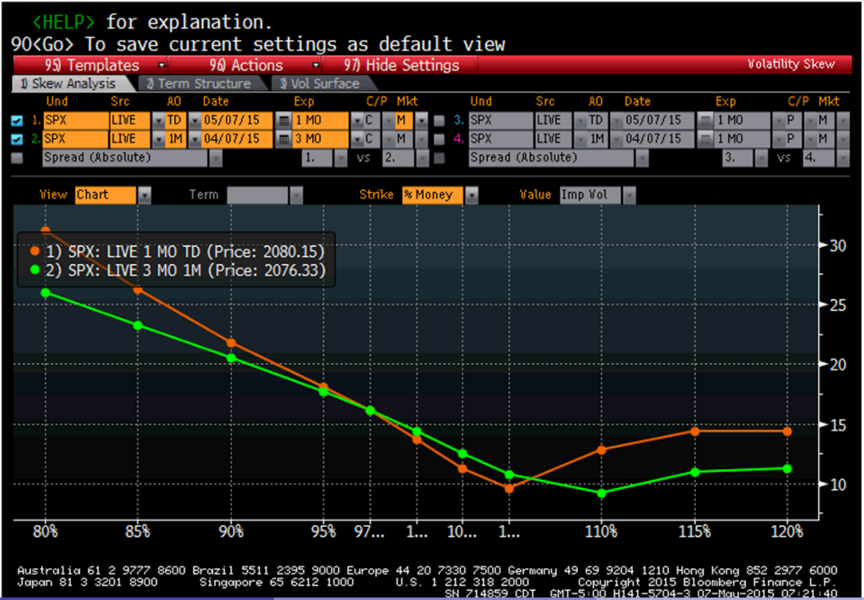
$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{v^2}{2}} dv =: \mathbb{P}\big(X \leq x\big)$$

- The Black-Scholes pricing formulas for European calls and puts

$$c(t, S_t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{25}$$

$$p(t, S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1) \tag{26}$$

Real World: Implied Volatilities



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	Und	Src	AO	Date	Exp	C/P	Mkt		Und	Src	AO	Date	Exp	C/P	Mkt
☑ 1.	SPX	LIVE	TD	05/07/15	1 MO	C	M	☐ 3.	SPX	LIVE	TD	05/07/15	1 MO	P	M
☑ 2.	SPX	LIVE	1M	04/07/15	3 MO	C	M	☐ 4.	SPX	LIVE	1M	04/07/15	1 MO	P	M
☐	Spread (Absolute)				1. vs 2.			☐	Spread (Absolute)				3. vs 4.		

View: Chart   Term: Strike: % Money   Value: Imp Vol

● 1) SPX: LIVE 1 MO TD (Price: 2080.15)
● 2) SPX: LIVE 3 MO 1M (Price: 2076.33)

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Final Words

So, depending on how you look at it, Quantitative Finance can be either practically incorrect, or incorrectly practical. That, in a nutshell, is the deadly ugliness and beauty of Quantitative Finance intertwined in *All is Vanity* (Ecclesiastes 1:2).