Answer. of Problem 1

The main idea is that when buying (selling) the base currency, buy (sell) at the ASK (BID) price. The other less obvious idea is that when buying the quote currency, you buy at 1/BID price, and when you sell the quote currency, you sell at 1/ASK price.

In the case of EUR/USD, euro is the base currency, since we are selling it, we obtain 1 mil × 1.12373 dollars. In the case of USD/JPY, the greenback is the base currency, and so we obtain 1 mil × 1.12373 × 101.310 yens.

In the case of GBP/JPY, Japanese yen is the quote currency. By selling the quote currency, you obtain 1 mil × 1.12373 × 101.310/131.612 pounds. In the case of GBP/USD, the Sterling is the base currency, and you get 1 mil × 1.12373 × (101.310/131.612) × 1.29687 dollars. Finallly, for EUR/USD, the greenback is the quote currency, and the amount of euros obtained is

$$\text{€}\,1\text{ mil} \times 1.12373 \times (101.310/131.612) \times 1.29687/1.12437 = \text{€}\,997{,}713.84.$$

The P&L is €997,713.84 − €1,000,000.00 = **−€2,286.16**.

But this is the academic answer. The answer in practice (real-world) is to take each trade one by one.

1. €1,000,000.00 × 1.12373 = $1,123,730.00.
2. $1,123,730.00 × 101.310 = ¥113,845,086.30
3. $\dfrac{\yen 113,845,086.30}{131.612}$ = £865,005.37
4. £865,005.37 × 1.29687 = $1,121,799.51
5. $\dfrac{\$1,121,799.51}{1.12437}$ = €997,713.84

It just happens that the rounding up and rounding down are offsetting each other and this answer is identical to the academic answer. ☐

Answer. of Problem 2

(A) Set up the payoff replication strategy:

$$26x + e^{0.02}y = 100$$
$$24x + e^{0.02}y = 10$$

By solving these linear equations, we obtain x = 45 shares, and the notional amount and the long (positive) or short (negative) direction of the bond is

$$y = e^{-0.02}(10 - 24 \times 45) = -\$1,048.81$$

Borrow $1,048.81 (issue a bond) and buy 45 shares at time 0.

(B) The cost of replication at time 0 is the present value or the price of this derivative.

$$25 \times 45 - \$1,048.81 = \mathbf{\$76.19}.$$

(C) Let p be the risk-neutral probability for the stock to go up.

$$\$100p + \$10(1-p) = \$76.19e^{0.02}$$

The solution is $1-p$ = **24.75%** for the stock to go down.

Another method that yields the same result is

$$\$26p + \$24(1-p) = \$25e^{0.02}.$$

These two methods are based on the first principle of $\mathbb{QF}$.

$\square$

Answer. of Problem 3

(A) To find the level of the term structure, set $T = 0$. We need to prove that

$$\lim_{T\to 0} \frac{1-e^{-T/\tau}}{\frac{T}{\tau}} = 1.$$

Apply L'Hôpital's rule, as it has the $0/0$ indeterminate form:

$$\lim_{T\to 0} \frac{1-e^{-T/\tau}}{\frac{T}{\tau}} = \lim_{T\to 0} \frac{\frac{1}{\tau}e^{-T/\tau}}{\frac{1}{\tau}} = 1$$

Therefore, the two terms in the parenthesis

$$\lim_{T\to 0} \left(\frac{1-e^{-T/\tau}}{\frac{T}{\tau}} - e^{-T/\tau} \right) = 1 - 1 = 0.$$

Hence, the level of the yield curve is r.

(B) The proof is a straightforward differentiation of Y_T with respect to T and a re-arrangement of terms.

$$Y_T' = \beta^{(l)} - \beta^{(s)} \left(e^{-T/\tau} \left[\frac{1}{T} + \frac{1}{\tau} \right] - \frac{\tau}{T^2} \left(1 - e^{-T/\tau} \right) \right).$$

(C) In considering the limit $T \to 0$, we need to focus on two terms in Y_T':

$$\frac{e^{-T/\tau}}{T} - \frac{\tau\left(1-e^{-T/\tau}\right)}{T^2},$$

which can be written as the $0/0$ indeterminate form when $T \to 0$:

$$f(T) := \frac{Te^{-T/\tau} - \tau\left(1 - e^{-T/\tau}\right)}{T^2}$$

L'Hôpital's rule is applicable, and we have

$$\lim_{T\to 0} f(T) = \lim_{T\to 0} \frac{e^{-T/\tau} - \dfrac{T}{\tau}e^{-T/\tau} - e^{-T/\tau}}{2T} = \lim_{T\to 0} \frac{-e^{-T/\tau}}{2\tau} = -\frac{1}{2\tau}.$$

Therefore,

$$\lim_{T\to 0} Y_T' = -\beta^{(s)}\left(\frac{1}{\tau} - \frac{1}{2\tau}\right) = \beta^{(l)} - \beta^{(s)}\frac{1}{2\tau}.$$

(D) It is clear that as $T \to \infty$, $e^{-T/\tau} \to 0$, and $1/T \to 0$. Hence $\lim_{T\to\infty} Y_T' = \beta^{(l)}$. This result allows us to interpret the parameter $\beta^{(l)}$ as the gradient of the long end part of the yield curve.

□

Answer. of Problem 4

The (annual) zero rates z_i are 1%, 1.2%, 1.3%, and 1.4% for $i = 1, 2, 3, 4$. Each i is a half year, i.e. $i = 2$ is a year. The discount factors are

$$\begin{aligned}
\mathrm{DF}_1 &= \frac{1}{(1+0.005)^1} = 0.9950248756218907\\
\mathrm{DF}_2 &= \frac{1}{(1+0.006)^2} = 0.9881071424336685\\
\mathrm{DF}_3 &= \frac{1}{(1+0.0065)^3} = 0.9807507802843708\\
\mathrm{DF}_4 &= \frac{1}{(1+0.007)^4} = 0.9724832231035931
\end{aligned}$$

(A) Let $n = 2$. we have

$$K = \frac{1 - \mathrm{DF}_2}{\mathrm{DF}_1 + \mathrm{DF}_2} = 0.5997\%.$$

Therefore the annualized swap rate is **1.1994%**.

(B) We need to solve for

$$\frac{c}{2}\mathrm{DF}_1 + \frac{c}{2}\mathrm{DF}_2 + \frac{c}{2}\mathrm{DF}_3 + \frac{c}{2}\mathrm{DF}_4 + 1\mathrm{DF}_4 = 1.$$

Hence

$$\frac{c}{2} = \frac{(1 - \mathrm{DF}_4)}{\sum_{i=1}^4 \mathrm{DF}_i}$$

Substituting in the values for the discount factors, we obtain c = **1.3981%** as the answer for the par rate corresponding to the tenor of 2 years.

□

Answer. of Problem 5

Take the data from Slide 28 of the lesson on Model-Free VIX. Given that the relevant (annual) risk-free rate is 0.5%, $S_t = 95.57$, and $T - t = 31/365$.

(A) The call struck at 90's midpoint is

$$c_t(90) = \frac{1}{2}\big(\$3.90 + \$8.30\big) = \$6.10.$$

From the put-call parity,

$$p_t(90) = \$6.10 + \$90e^{-0.005\times 31/365} - \$95.57 = \mathbf{\$0.49}.$$

Likewise

$$c_t(92.5) = \frac{1}{2}\big(\$1.70 + \$6.10\big) = \$3.90$$

From the put-call parity, we obtain

$$p_t(92.5) = \$3.90 + \$92.5e^{-0.005\times 31/365} - \$95.57 = \mathbf{\$0.79}.$$

Note that $p_t(92.5) > p_t(90)$, which is the monotonicity property of the price curve for a put.

(B) The midpoint of the ITM put option struck at 100 is

$$p_t(100) = \frac{1}{2}\big(\$2.50 + \$6.50\big) = \$4.50$$

Applying put-call parity,

$$c_t(100) = \$4.50 + \$95.57 - \$100e^{-0.005\times 31/365} = \mathbf{\$0.11}.$$

Likewise, the midpoint of the ITM put option struck at 102.5 is

$$p_t(102.5) = \frac{1}{2}\big(\$4.70 + \$9.10\big) = \$6.90$$

Applying the put-call parity,

$$c_t(102.5) = \$6.90 + \$95.57 - \$102.5e^{-0.005\times 31/365} = \mathbf{\$0.01}.$$

Note that $c_t(102.5) < c_t(100)$, which is the monotonicity property of the price curve for a call.

□

Answer. of Problem 6

The static replication of a payoff function $f(S)$ at maturity is

$$f(S_T) = f(\lambda) + f'(\lambda)(S_T - \lambda) + \int_0^{\lambda} f''(K)(K - S_T)^+ dK + \int_{\lambda}^{\infty} f''(K)(S_T - K)^+ dK.$$

(A) Note that in this context, λ is a known quantity at time 0. Therefore, the first term $f(\lambda)$ is a sure cash flow and thus no risk. So its PV at time 0 is to discount it by $e^{-r_0 T}$ where r_0 is the risk-free rate.

The second term is the payoff of a long forward position. The number of forward contracts is $f'(\lambda)$. Each contract can be replicated by a long position in a call, and a short position in a put. Both options have the same strike price of λ.

The integrand in the third term is the payoff of a portfolio of put options at expiration T. Each put option at time 0 has the price of $p_0(K)$. The number of contracts is $f''(K)$ for $p_0(K)$ struck at K.

The integrand in the last term is the payoff of a portfolio of call options at expiration T. Its price (PV) at time 0 is $c_0(K)$. The number of contracts is $f''(K)$ for $c_0(K)$ struck at K.

Static replication of these European-style contracts means that no further transaction is needed once the discount bond and the option positions are established. All these replications follow the first principle of $\mathbb{Q}$F.

(B) The payoff function is $f(S_T) = \dfrac{1}{S_0^2} S_T^2$, with S_T being the variable and S_0 a constant since it is observable at time 0. Now,

$$f'(S_T) = \frac{2}{S_0^2} S_T, \qquad \text{and} \qquad f''(S_T) = \frac{2}{S_0^2}.$$

We let $\lambda = F_0$, the forward price, which is $S_0 e^{r_0 T}$. Given $r_0 = 1\%$ and $S_0 = \$6$,

$$F_0 =_0 e^{r_0 T} = \$6 e^{0.01} = \$6.06.$$

Therefore, the price of the discount bond is, given $T = 1$ year,

$$e^{-r_0 T} \left(\frac{F_0}{S_0} \right)^2 = e^{-r_0 T} e^{2 r_0 T} = e^{r_0 T} = e^{r_0} = \$ e^{0.01} = \$1.01$$

Next, the number of forward contract is

$$f'(F_0) = \frac{2}{S_0^2} F_0 = \frac{2 e^{r_0 T}}{S_0} = \frac{2 \times 1.01}{6} = \frac{1.01}{3}.$$

Theoretically, by put-call parity, $c_0(F_0) - p_0(F_0) = 0$ if the strike price is exactly equal to the forward price. If that is the case, the forward contract's PV is zero. However, it is more likely that F_0 does not equal to any strike price of an option chain. We choose the nearest $c_0(6) - p_0(6)$ to replicate a contract of forward. Hence the cost is

$$\frac{1.01}{3}\big(c_0(6) - p_0(6)\big).$$

We need to discretize the integral as a Riemann sum. The strike interval is $\Delta K = 0.25$, and

$$\frac{2}{S_0^2}\int_0^{F_0} p_0(K)\,dK \approx \frac{2}{S_0^2}\sum_{i=1}^{I} p_0(K_i)\Delta K,$$

where $K_1 = 4$ and $K_I = 6$. Therefore, the PV involving put options is, since $\Delta K = 0.25$,

$$\frac{2\times 0.25}{36}\big(p_0(4) + p_0(4.25) + \cdots + p_0(5.75) + p_0(6)\big).$$

Likewise, the PV involving call options is

$$\frac{2}{S_0^2}\int_{F_0}^{\infty} c_0(K)\,dK \approx \frac{2}{S_0^2}\sum_{i=I}^{J} c_0(K_i)\Delta K,$$

where $K_I = 6$ and $K_J = 8$. Accordingly, the PV is

$$\frac{2\times 0.25}{36}\big(c_0(6) + c_0(6.25) + \cdots + c_0(7.75) + c_0(8)\big).$$

The minimum price (total PV) is, for one contract of obtaining the payoff $\left(\frac{S_T}{S_0}\right)^2$,

$$\begin{aligned}\text{PV} =& 1.01 + \frac{1.01}{3}\big(c_0(6) - p_0(6)\big)\\ &+ \frac{1}{72}\big(p_0(4) + p_0(4.25) + \cdots + p_0(5.75) + p_0(6) + c_0(6) + c_0(6.25) + \cdots + c_0(8)\big).\end{aligned}$$

The bank will add a fee to this minimum PV.

□

Remark: In practice, since there is no strike corresponding to $F_0 = 6.06$, the quant need to choose between $K = 6$ or $K = 6.25$. Here $K = 6$ is chosen. In practice, quants will consider the prices of $p_0(6) + c_0(6)$ versus $p_0(6.25) + c_0(6.25)$. They will choose the one that has a higher premium. Also, for the forward contract, they will look at $c_0(6) - p_0(6)$ versus $c_0(6.25) - p_0(6.25)$. Again, they will choose the one that has a higher value. The main consideration is that the replicated payoff must not be less than the payoff $f(S_T)$.