Options

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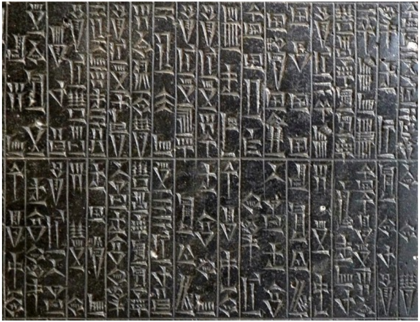
Code of King Hammurabi, 1792 to 1750 BC

Picture source: Kept in the Louvre



Picture source: Code of Hammurabi

King Hammurabi's 48-th Code

If any one owe a debt for a loan, and a storm prostrates the grain, or the harvest fail, or the grain does not grow for lack of water; in that year he need not give his creditor any grain, he washes his debt-tablet in water and pays no rent for this year.

From the perspective of a creditor,

- → Underlying asset: grain
- → Expiration: at harvest
- → Delivery mode: physical
- → Condition: if not (a storm prostrates the grain, or the harvest fail, or the grain does not grow for lack of water)

Discussion

→ Is the 48-th Code fair to the farmer?

→ Is the 48-th Code fair to the creditor?

→ Why should the creditor lend to the farmer in the first place?

Publicly Listed Options

- → Options used to be, and still is traded OTC.
- → Chicago Board of Options Exchange (CBOE) for publicly listed options
- → Black-Scholes' pricing formulas
- → 1973: 1.1 million option contracts on 32 equity issues
- → 2014: 3.8 billion contracts on 4,278 issues

Basic Option Terminology

- ↔ Strike price K: the equivalent of forward price in the forward contract
- ↔ Plain vanilla: standard type of option, one with a simple expiration date and strike price and no additional feature
- ↔ Call option: right (obligation) of an option buyer (seller) to buy (sell) the underlying asset at the strike price
- ↔ Put option: right (obligation) of an option buyer (seller) to sell (buy) the underlying asset at the strike price
- ↔ European: buyers can exercise the right to buy or sell at expiration date only
- ↔ American: buyers can exercise the right to buy or sell at or before expiration

Concept Checker: Odd One Out?

1. For European option buyers, they must hold the option to maturity.
2. For European option sellers, they must hold the option to maturity.
3. At expiration, American options is worthless.
4. Any option seller can exit the obligation by buying back the options sold in the marketplace.

Moneyness of Call Options

- $\leftrightarrow$ In-the-money call option: strike price is lower than the underlying price
- $\leftrightarrow$ At-the-money call option: strike price is equal to the underlying price
- $\leftrightarrow$ Out-of-the-money call option: strike price is higher than the underlying price

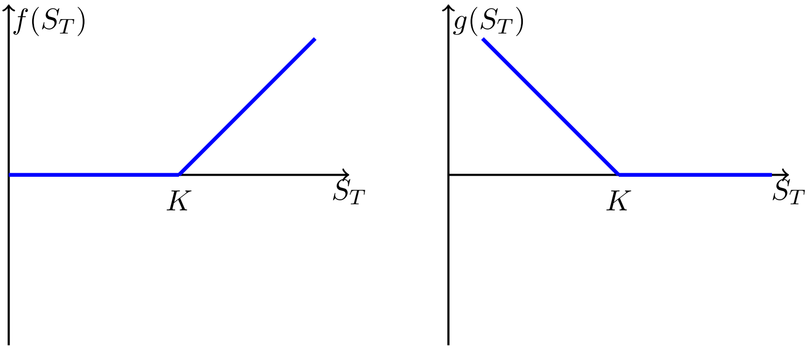
Moneyness of Put Options

- ↔ In-the-money put option: strike price is higher than the underlying price
- ↔ At-the-money put option: strike price is equal to the underlying price
- ↔ Out-of-the-money put option: strike price is lower than the underlying price
- ↔ DOOM option: Deep out-of-the-money option

Option Price

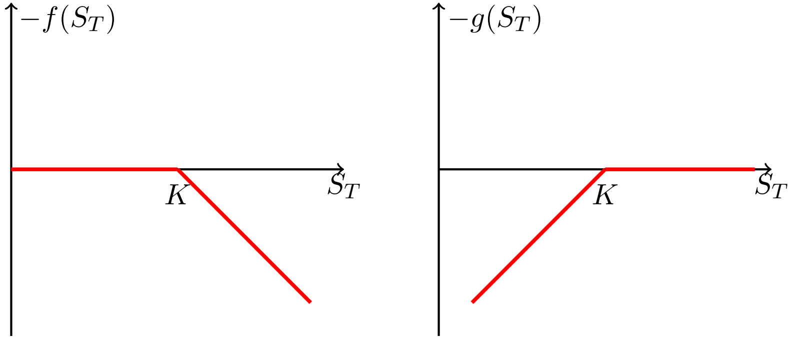
- Because of the asymmetry of right and obligation, option sellers demand a premium from option buyer.
- The premium is also known as the option price.
- One option contract in the U.S. typically allows the call (put) option buyer to exercise the right to buy (sell) 100 shares at the strike price K.
- But how should one go about pricing an option?

Buyers' Payoff Functions of Plain Vanilla Options



The payoffs of plain vanilla call (left) and put (right) options buyer at maturity T when the underlying price is S_T.

Sellers' Payoff Functions of Plain Vanilla Options



The payoffs of plain vanilla call (left) and put (right) options seller at maturity T when the underlying price is S_T..

Quiz: Learn to Think Geometrically

(1) What is the value of $f(S_T) - g(S_T)$?

Answer: ______________________________

(2) What is the value of $g(S_T) - f(S_T)$?

Answer: ______________________________

(3) What is the functional form of call's payoff?

Answer: ______________________________

(4) What is the functional form of put's payoff?

Answer: ______________________________

Option Chain (of Strike Prices)

Underlying stock: VALE Vale SA

Last: 4.85 Change: $0.90 % Change: 22.63% as of Jun 06 2016, 3:55 PM ET

Options Expiration: Sep 15, 2016

Call					Strike	Put				
Interest	Volume	Last	Bid	Ask	Price	Bid	Ask	Last	Volume	Interest
0	0	0.00	4.10	4.35	**0.5**	0.00	0.02	0.00	0	0
4,357	0	3.30	3.60	3.85	**1**	0.00	0.03	0.02	0	607
40	0	1.28	3.10	3.35	**1.5**	0.00	0.03	0.19	0	400
1,069	0	2.32	2.60	2.83	**2**	0.01	0.04	0.03	0	1,181
6	0	1.85	2.14	2.36	**2.5**	0.04	0.07	0.11	0	13,254
5,167	0	1.25	1.70	1.78	**3**	0.10	0.14	0.15	0	1,911
281	0	1.20	1.30	1.39	**3.5**	0.18	0.20	0.40	1	8,704
1,703	0	0.95	0.96	1.05	**4**	0.36	0.40	0.40	0	2,094
11,908	228	0.67	0.76	0.85	**4.5**	0.57	0.62	0.63	20	4,087
33,222	5,798	0.48	0.56	0.60	**5**	0.75	0.80	0.92	5,087	30,836
4,817	259	0.34	0.38	0.40	**5.5**	1.15	1.23	1.25	0	6,949
2,481	0	0.20	0.19	0.24	**6**	1.54	1.62	1.82	47	7,230
25,464	3	0.06	0.09	0.11	**7**	2.28	2.38	3.07	14	18,960
6,171	20	0.03	0.04	0.06	**8**	3.35	3.50	3.85	0	141
214	0	0.15	0.00	0.04	**9**	4.20	4.45	4.80	0	300
0	0	0.00	0.00	0.03	**10**	5.20	5.45	5.35	0	77
1	0	0.05	0.00	0.03	**11**	6.20	6.45	0.00	0	0
0	0	0.00	0.00	0.03	**12**	7.20	7.45	0.00	0	0

Source: Optionetics

$\leftrightarrow$ Options are not traded in isolation.

Option Market Quotes and Volume

- $\leftrightarrow$ Bid: the price market is currently willing to pay for buying an option
- $\leftrightarrow$ Ask: the price at which market will sell an option
- $\leftrightarrow$ Last: the price at which an option was traded most recently
- $\leftrightarrow$ Volume: number of contracts traded during the trading session
- $\leftrightarrow$ (Open) Interest: number of open contracts

Intrinsic Value

- ↔ The dollar amount an option is in the money.
- ↔ Out-of-the-money options have no intrinsic value.
- ↔ What about at-the-money options?

  Answer: ______________________________

- ↔ Example: Strike price $4.50 versus underlying price of $4.85.
  - Intrinsic value of call option = $\$4.85 - \$4.50 = \$0.35$.
  - Intrinsic value of put option = 0

Time Value

- ↔ For the call option, the intrinsic value is only \$0.35 whereas the midpoint of its bid and ask prices is \$0.805. Why is the call option priced at a higher price?
- ↔ The difference of \$0.805 - \$0.35 = \$0.455 is called the option's time value.
- ↔ Why would a call option buyer pay for the time value?
- ↔ For that matter, why would the put option of zero intrinsic value is worth $(\$0.57 + \$0.62)/2 = \$0.595$?

Concept Checker: Odd One Out?

1. Option holders have rights but sellers have obligations.
2. Only in-the-money options have intrinsic value.
3. All options have time value but not intrinsic value.
4. The time value of option is always bigger than the intrinsic value.

Put and Call

- Consider a put p_t and a call c_t on the same underlying asset S_t, maturity T, and strike K. Of these 5 quantities, only T is not a price. Assume that one option contract is to one unit of the underlying asset.

- In addition, suppose lending and borrowing can be done at the interest rate of r_0 for tenor T with continuous compounding

- Positions at time $t = 0$
  - Borrow and amount $Ke^{-r_0 T}$
  - Sell a call c_0
  - Use the borrowed fund and proceeds to buy the stock at price S_0, and a put p_t.

- Total cash flow at time 0: $Ke^{-r_0 T} + c_0 - S_0 - p_0$

Payoffs of Long Stock, Short Call, and Long Put

At maturity $t = T$, if $S_T < K$, then

- The call option is worthless.
- Exercise the put option to sell the asset at the strike price K.
- Pay the debt of amount K

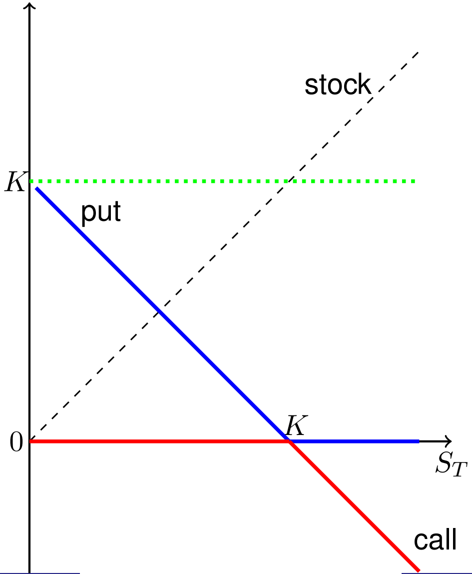
If $S_T > K$, then

- The put option is worthless.
- The call option holder will exercise and you have to sell the asset at the strike price K.
- Pay the debt of amount K

If $S_T = K$, then

- Both options are not exercised
- Selling the underlying asset at S_T in the spot market.
- Pay the debt of amount K

Payoff Diagram



Application of the Principles of $\mathbb{Q}$F

- The net cash flow at time T is zero, regardless of the outcomes (either $S_T < K$ or $S_T > K$ or $S_T = K$).
- By the first principle of $\mathbb{Q}$F, the cash flow at time 0 must also be zero because there is no uncertainty and hence no risk. Why no uncertainty? All the prices and the interest rate are known at time 0!
- Hence

$$Ke^{-rT} + c_0 - S_0 - p_0 = 0.$$

and this put-call parity is more commonly written as

$$c_0 - p_0 = S_0 - Ke^{-r_0 T}.$$

- At time t, it is written as

$$\boxed{c_t - p_t = S_t - Ke^{-r_t(T-t)}} \tag{1}$$

First Application of Put-Call Parity

- With respect to the intersection point, the put-call parity (1) becomes

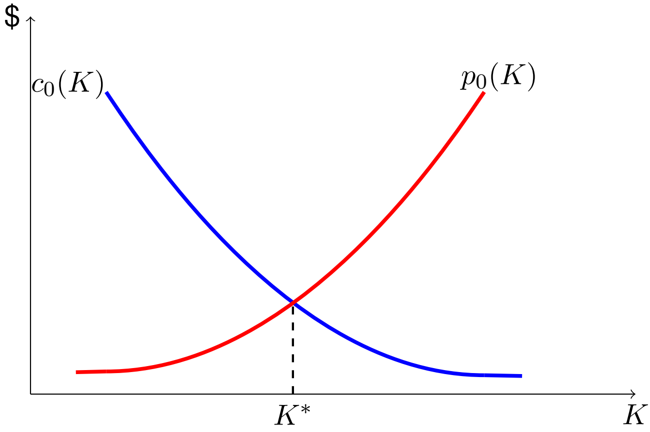
$$0 = S_t - K^* e^{-r_t(T-t)},$$

and it is re-written as

$$K^* = S_t\, e^{r_t(T-t)}.$$

- This result shows that the strike price K^* at which the call option price equals the put option price is none other than the forward price of the underlying asset at time t.
- K^* is called the implied forward price.

Option Price Curves as Functions of Strike K



Capital Structure

- The composition of equity and bond by which a corporation finances its assets is called the capital structure.
- A firm that has zero debt is said to be unlevered, whereas a firm that also issues corporate bond in addition to equity is said to be levered.
- Suppose a levered firm has outstanding only a zero-coupon bond with face value K maturing at time T.
- Next, we denote the market value of an unlevered but otherwise identical firm by S_t.

Modigliani-Miller Proposition I

- It does not matter what capital structure a company uses to finance the operation.
- A deeper look at the root of the Modigliani-Miller Proposition I finds that it is grounded on the third principle of $\mathbb{Q}$F.
- If a firm's market value could be changed by changing the proportion of stocks and bonds they issue, then arbitrageurs could also repackage the existing stocks and bonds to make a sure profit.
- Hence, the value of the firm should depend only on the sum of the values of its stocks and bonds, not on whether the firm's capital is weighted more heavily to debt or equity.

Second Application: Derivation of MM Proposition I

- The capital structure of a firm, i.e., whether levered or unlevered, is irrelevant to the market valuation of a firm.
- The put-call parity (1) can also be expressed as, at time 0,

$$S_0 = c_0 + Ke^{-r_0 T} - p_0. \tag{2}$$

- The actual market value of the debt need to be discounted to reflect credit risk so that it is worth only $Ke^{-r_0 T} - p_0$.
- The put premium p_0 may be interpreted as the insurance premium required by the bondholders against default by the levered firm.
- What about the interpretation of c_0?

Tutorial

- Define an interest rate $\grave{r}$ by

$$Ke^{-\grave{r}_0 T} := Ke^{-r_0 T} - p_0,$$

Show that the spread $\grave{r}_0 - r_0$ can be well approximated by

$$\grave{r}_0 - r_0 = \frac{p_0}{KT}$$

- Proof

We rewrite the definition as

$$K\left(e^{-r_0 T} - e^{-\grave{r}_0 T}\right) = p_0$$

At the first order of the expansion, we have

$$-r_0 T - (-\grave{r}_0 T) = \frac{p_0}{K} \qquad \Longrightarrow \qquad \grave{r}_0 - r_0 = \frac{p_0}{KT}$$

Monotonous with Respect to the Strike Price

- Consider two strike prices K_1 and K_2 with $K_1 < K_2$, and also the following two portfolios:
  - Portfolio 1
    A European call option $c_0(K_1)$ of strike K_1
  - Portfolio 2
    A European call option $c_0(K_2)$ of strike K_2
- At expiration, the value of portfolio 1 (denoted by $V_1(T)$) is by definition $\max(S_T - K_1, 0)$, whereas portfolio 2's value (denoted by $V_2(T)$) is $\max(S_T - K_2, 0)$.
- With respect to these two strike prices, and the underlying asset price at time T, denoted by S_T, three mutually exclusive scenarios are exhaustive:
  1. $S_T < K_1$
  2. $K_1 \leq S_T \leq K_2$
  3. $S_T > K_2$.

Monotonous with Respect to the Strike Price (Cont'd)

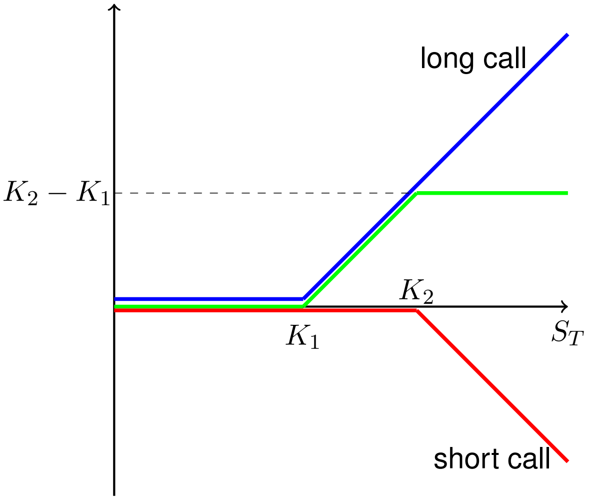
- If $S_T < K_1$, both portfolios are worthless, and $V_2(T) = V_1(T) = 0$.
- If $K_1 \leq S_T \leq K_2$, the value of portfolio 1 is $S_T - K_1$ but the value of portfolio 2 is still zero. So $V_2(T) < V_1(T)$ in the second scenario.
- In the third scenario, both options are in the money, $V_1(T) = S_T - K_1$ and $V_2(T) = S_T - K_2$. Since $K_1 < K_2$, it it clear that $S_T - K_2 < S_T - K_1$, which implies that $V_2(T) < V_1(T)$ in the third scenario as well.
- Accordingly, $V_2(T) \leq V_1(T)$ at time T. By the third principle of $\mathbb{Q}$F, it must be that at time 0,

$$c_0(K_2) \leq c_0(K_1).$$

Lower Bound for the Slope

- What is the lower bound for the slope of the call option price curve $c_0(K)$?
- Portfolio 3: Bull Call Spread
  A long position in call option $c_0(K_1)$ of strike price K_1 and a short position in call option $c_0(K_2)$ of strike price K_2
- Portfolio 4
  A risk-free time deposit of amount $(K_2 - K_1)e^{-rT}$, tenor T, and interest rate r
- The marked-to-market value of portfolio 3 at time 0 is $V_3(0) = c_0(K_1) - c_0(K_2)$, which is non-negative as proven earlier.

Bull Call Spread's Payoff Diagram



Bounds for Call Option's Gradient

- At expiration, it is evident that the payoff of the bull call spread is less than or equal to $K_2 - K_1$, which coincides with the value of portfolio 4.
- Since $V_3(T) \leq V_4(T)$ at time T, their prices at time 0, according to the third principle of $\mathbb{Q}$F, must observe the same inequality:

$$c_0(K_1) - c_0(K_2) \leq (K_2 - K_1)e^{-rT} \leq K_2 - K_1.$$

- In conjunction with the earlier upper bound result that $c_0(K_2) - c_0(K_1) \leq 0$, these inequalities provide an upper bound and a lower bound for the gradient of the call price curve $c_0(K)$ as follows:

$$-1 \leq \frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq 0.$$

In summary, $c_0(K)$ is a downward sloping curve.

Linear Interpolation

- Consider three strike prices sorted by magnitude as $K_1 < K_2 < K_3$.
- Moreover, we define a positive ratio as follows:

$$\lambda := \frac{K_3 - K_2}{K_3 - K_1} < 1. \quad (3)$$

- This ratio is smaller than one and thereby allows us to express K_2 as a linear combination of K_1 and K_3:

$$K_2 = \lambda K_1 + (1 - \lambda)K_3.$$

Portfolios for Convexity Analysis

- Portfolio 5
  A long position in one contract of call option struck at K_2

- Portfolio 6
  A long position in λ contracts of call option struck at K_1 and another long position in $1 - \lambda$ contracts of call option of strike price K_3

Payoff Scenarios

Scenario	$S_T < K_1$	$K_1 \leq S_T \leq K_2$	$K_2 < S_T \leq K_3$	$S_T > K_3$
Portfolio 5 ♦ Long 1 $c_0(K_2)$	0	0	$S_T - K_2$	$S_T - K_2$
Portfolio 6 ♦ Long λ $c_0(K_1)$ ♦ Long $1 - \lambda$ $c_0(K_3)$	0 0	$\lambda(S_T - K_1)$ 0	$\lambda(S_T - K_1)$ 0	$\lambda(S_T - K_1)$ $(1-\lambda)(S_T - K_3)$
Aggregate of Portfolio 6	0	$\lambda(S_T - K_1)$	$\lambda(S_T - K_1)$	$S_T - K_2$

Clearly, at maturity, $V_5(T) \leq V_6(T)$ in all scenarios.

Curvature of the Call Option Price Function

- Again, the third principle of $\mathbb{Q}$F requires the prices of these two portfolios at time 0 to be

$$c_0(K_2) \leq \lambda c_0(K_1) + (1-\lambda)c_0(K_3).$$

- Substituting in the explicit form of λ, i.e., (3) into this inequality, we arrive at

$$(K_3 - K_1)c_0(K_2) \leq (K_3 - K_2)c_0(K_1) + (K_2 - K_1)c_0(K_3).$$

- We write the left side of the inequality as $(K_3 - K_2)c_0(K_2) + (K_2 - K_1)c_0(K_2)$, and then rearrange the inequality into

$$(K_3 - K_2)\big(c_0(K_2) - c_0(K_1)\big) \leq (K_2 - K_1)\big(c_0(K_3) - c_0(K_2)\big).$$

- Next, we divide both sides by $(K_3 - K_2)(K_2 - K_1)$ to obtain

$$\frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq \frac{c_0(K_3) - c_0(K_2)}{K_3 - K_2}.$$

Monotonicity, Gradient Boundedness, and Convexity

- $K_1 < K_2 < K_3$
- Monotonicity in the option price level

$$c_0(K_2) \leq c_0(K_1); \qquad p_0(K_1) \leq p_0(K_2).$$

- Boundedness in the gradient

$$-1 \leq \frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq 0; \qquad 0 \leq \frac{p_0(K_2) - p_0(K_1)}{K_2 - K_1} \leq 1.$$

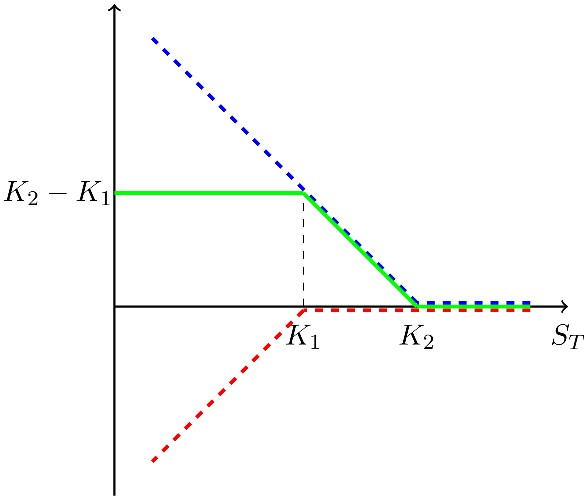
- Convexity

$$\frac{c_0(K_2) - c_0(K_1)}{K_2 - K_1} \leq \frac{c_0(K_3) - c_0(K_2)}{K_3 - K_2}; \quad \frac{p_0(K_2) - p_0(K_1)}{K_2 - K_1} \leq \frac{p_0(K_3) - p_0(K_2)}{K_3 - K_2}.$$

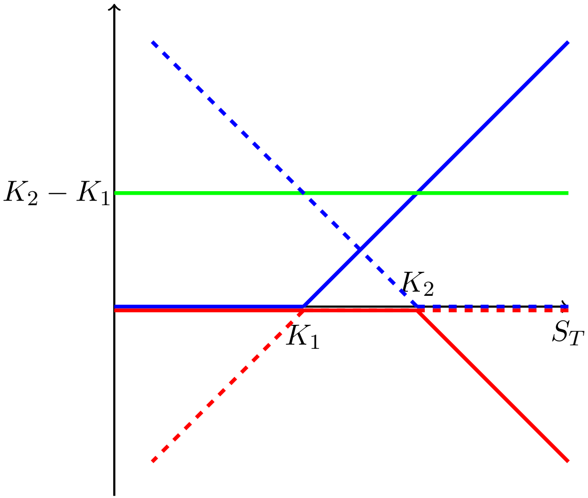
Model-Free

- It is important to recognize that the three features of monotonicity, gradient boundedness, and convexity are model-free in the following sense:
  - Type 1: Models for the stochastic dynamics of the underlying asset are not needed.
  - Type 2: Models to price the options are not needed.
- Is put-call parity model-free? Which Type?

Payoff Diagram of a Bear Put Spread



Payoff Diagram of a Box Spread



Box Spread Cash Flows

Asset price range	Payoff from bull call spread	Payoff from bear put spread	Total payoff
$S_T < K_1$	0	$K_2 - K_1$	$K_2 - K_1$
$K_1 \le S_T \le K_2$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$
$S_T > K_2$	$K_2 - K_1$	0	$K_2 - K_1$

- The payoff of a long position in the bull call spread and a long position in the bear put spread is a constant $K_2 - K_1$!
- By the first principle of $\mathbb{Q}$F, the value of a long position in the box spread must be

$$(K_2 - K_1)e^{-r_0 T}$$

- Accordingly, we have

$$\boxed{c_0(K_1) - c_0(K_2) + p_0(K_2) - p_0(K_1) = (K_2 - K_1)e^{-r_0 T}.}$$

Takeaways

- Plain vanilla put and call options: exercise styles, maturities, strike prices, settlement modes
- Intrinsic value versus time value
- Model-free put-call parity
- Modigliani-Miller Proposition I: Irrelevance of capital structure of a firm
- Monotonicity, gradient boundedness, and convexity of option price curves
- Bull call spread versus bear put spread

Week 6 Assignment from Chapter 6

- Q1
- Q6

Week 6 Additional Exercises

1. The delta of an option is the first-order partial derivative of the option price.
   (A) Applying the put-call parity, show that the deltas of a put and a call, denoted by Δ_p and Δ_c, respectively, must satisfy

   $$\Delta_c - \Delta_p = 1$$

   (B) Consequently, the second-order derivative of the option price—Gamma—must be

   $$\Gamma_c = \Gamma_p$$

2. In Slide 37, under the scenario where $K_2 < S_T \leq K_3$, show that $S_T - K_2$ indeed is less than $\lambda\big(S_T - K_1\big)$.