Taylor's Expansion

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Taylor's Theorem

On Taylor's Theorem:

> *...le principal fondement du calcul différentiel...*
>
> — Joseph-Louis Lagrange (1736–1813)

Source: Biographie universelle des musiciens et bibliographie générale de la musique: Sa - Zy, Volume 8, Page 337.

What is Taylor's Theorem?

A differentiable function may be represented by **Taylor's series expansion** of the function $f(x)$ about a finite real number a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$\cdots + \frac{f^{(n)}(\lambda)}{n!}(x-a)^n + \cdots$$

Taylor's Principle and Practice

Source: Brook Taylor

The true and best way of learning any Art, is not to see a great many Examples done by another Person, but to possess ones self first of the Principles of it, and then to make them familiar, by exercising ones self in the Practice.

Source: New Principles of Linear Perspective, Page vi

Recalling Your Pre-U Math!

→ When $a = 0$, we have the Maclaurin series expansion!

$$\begin{aligned} f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0) \end{aligned}$$

→ Two special cases are very useful in $\mathbb{Q}$uantitative Finance

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for all } x$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

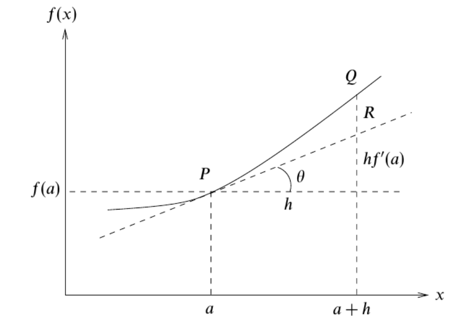
for $-1 < x \leq 1$

Geometric Intuition for Approximation

$$\int_a^{a+h} f'(x)dx = f(a+h) - f(a)$$

$$\implies \quad f(a+h) = f(a) + \int_a^{a+h} f'(x)dx \tag{1}$$

- The first-order Taylor series approximation of $f(x)$ is the slope of the function at P, i.e. $\tan\theta = f'(a)$.
- The value of the function at Q, $f(a+h)$, is approximated by the ordinate of R, $f(a) + hf'(a)$.



First-Order Approximation

- As a first-order approximation, if h is sufficiently close to a, we approximate $f'(x)$ as $f'(a)$ in (1). Then,

$$f(a+h) \approx f(a) + hf'(a).$$

- Noting that $x - a = h$, we have

$$f(x) \approx f(a) + (x-a)f'(a).$$

- If $f(x)$ is at least twice differentiable, then

$$f'(x) \approx f'(a) + (x-a)f''(a),$$

Second-Order Approximation

→ The second-order approximation is

$$f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2}h^2.$$

→ Proof: Substitute $f'(x) \approx f'(a) + (x-a)f''(a)$ into the integral in (1)

$$\begin{aligned} f(a+h) &\approx f(a) + \int_a^{a+h} \big(f'(a) + (x-a)f''(a)\big)\, dx \\ &\approx f(a) + f'(a)x\Big|_a^{a+h} + \frac{1}{2}f''(a)(x-a)^2\Big|_a^{a+h} \\ &\approx f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 \end{aligned}$$

Taylor's Expansion

→ Assumption: The function $f(x)$ is expressible as a polynomial:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots ,$$

where a is a constant, and we want to find c_i for $i = 0, 1, \ldots, \infty$.

→ To find c_0, set $x = a$. Then all $x - a$ terms become zero. Accordingly,

$$c_0 = f(a).$$

→ To find c_1, take the derivative of $f(x)$, resulting in

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

→ Again, set $x = a$. All $x - a$ terms vanish and we have

$$c_1 = f'(a).$$

Taylor's Expansion (cont'd)

→ To obtain c_2, take the second derivative:

$$f''(x) = 2c_2 + 3 \times 2 \times c_3(x-a) + \cdots$$

With $x = a$, we obtain

$$c_2 = \frac{1}{2} f''(a).$$

→ To obtain c_3, take the third derivative:

$$f'''(x) = 3 \times 2c_3 + 4 \times 3 \times 2c_4(x-a) + \cdots$$

With $x = a$, we have

$$c_3 = \frac{1}{3 \times 2} f'''(a).$$

Taylor's Expansion (cont'd)

→ In general

$$f^{(n)}(x) = n! \times c_n + (n+1) \times n! \times c_{n+1}(x-a) + \cdots .$$

By setting $x = a$, we have

$$c_n = \frac{1}{n!} f^{(n)}(a).$$

→ Thus, the Pre-U math is enough to obtain Taylor's expansion!

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j \qquad (2) \end{aligned}$$

Takeaway

source

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$