# Taylor's Theorem

**Christopher Ting**

http://www.mysmu.edu/faculty/christophert/

✉: christopherting@smu.edu.sg

☎: 6828 0364

👤: LKCSB 5036

# Intuition

- We want to approximate a function $f$ near a point $a$ by a polynomial $P_n(x)$ of degree $n$.
- For $n = 0$, the best (constant) approximation near $a$ is, for all $x$,
$$P_0(x) = f(a).$$
- For $n = 1$, the best (linear) approximation near $a$ is
$$P_1(x) = f(a) + f'(a)x.$$
- For $n = 2$, the best (quadratic) approximation near $a$ is
$$P_2(x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2.$$

# Definition of Taylor's Polynomial

- Let $\alpha < \beta$ be real numbers.
- Let $n \geq 0$ be an integer.
- Let $a \in [\alpha, \beta]$.
- Suppose that $f : [\alpha, \beta] \to \Re$ is $n$ times differentiable at $a$.
- The polynomial $P_n(h) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k$ is called the **Taylor polynomial** of order $n$ for $f$ at $a$.
- When $h = x - a$, $n = \infty$, and $f$ is smooth (infinitely differentiable), then $f$, i.e.,

  $$P_\infty(x-a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

  is called the **Taylor series** of $f$ centered at $a$.
- Big question: does $P_\infty(x-a)$ equal to $f(x)$?

# Lemma

Define $g(h) := f(a+h) - P_n(h)$. Then $g^{(j)}(0) = 0$ for $j = 0, 1, \dots, n$.

- Example

$$\begin{aligned} P_n^{(3)}(h) &= \frac{f^{(3)}(a)}{3!} 3 \times 2 \times 1 + \frac{f^{(4)}(a)}{4!} 4 \times 3 \times 2h \\ &\quad + \cdots + \frac{f^{(n)}(a)}{n!} n(n-1)(n-2)h^{n-3} \end{aligned}$$

Hence, at $h = 0$, we have $P_n^{(3)}(0) = f^{(3)}(a)$.

- Proof: In general,

$$\begin{aligned} P_n^{(j)}(h) &= \frac{f^{(j)}(a)}{j!} j! + \frac{f^{(j+1)}(a)}{(j+1)!} ((j+1)j \cdots 2)h + \cdots \\ &\quad + \frac{f^{(n)}(a)}{n!} (n(n-1) \cdots (n-j+1))h^{n-j}. \end{aligned}$$

Thus, $g^{(j)}(0) = f^{(j)}(a) - P_n^{(j)}(0) = 0$ holds.

# Approximation Theorem

$$\lim_{h\to 0}\frac{f(a+h)-P_n(h)}{h^n}=0$$

Proof

- Let $g(h) := f(a+h) - P_n(h)$ as before.
- Note that $P^{(n-1)}(h) = f^{(n-1)}(a) + f^{(n)}(a)h$.
- Apply L'Hôpital's rule $n-1$ times:

$$\begin{aligned} n!\lim_{h\to 0}\frac{g(h)}{h^n} &= \lim_{h\to 0}\frac{g^{(n-1)}(h)}{h} \\ &= \lim_{h\to 0}\frac{f^{(n-1)}(a+h)-f^{(n-1)}(a)-f^{(n)}(a)h}{h} \\ &= \lim_{h\to 0}\frac{f^{(n-1)}(a+h)-f^{(n-1)}(a)}{h}-f^{(n)}(a)=0. \end{aligned}$$

# Concept Checkers

- What is the intuitive meaning of $\lim\limits_{h\to 0} \dfrac{f(a+h) - P_n(h)}{h^n} = 0$?

- Since $g^{(n)}(0) = 0$ as shown in the lemma, why can't we apply L'Hôpital's rule $n$ times to obtain

$$n! \lim_{h\to 0} \frac{g(h)}{h^n} = \lim_{h\to 0} f^{(n)}(a+h) - f^{(n)}(a)$$

in proving the approximation theorem?

# Taylor's Theorem

- Suppose $f^{(k)}$ is continuous on $[\alpha, \beta]$ for $0 \leq k \leq n$
- Suppose $f^{(n+1)}(x)$ exists and is finite for all $x \in (\alpha, \beta)$.
- For $\alpha - a \leq h \leq \beta - a$, define the **remainder** as

$$R_n(h) := f(a+h) - P_n(h). \tag{1}$$

- Then for each fixed $h \neq 0$ and each integer $u < n+1$, there exists $0 < \theta < 1$ such that

$$R_n(h) = \frac{h^{n+1}(1-\theta)^u}{(n+1-u)n!} f^{(n+1)}(a+\theta h). \tag{2}$$

- Consequently, with $x = a + h$,

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x-a). \end{aligned} \tag{3}$$

# Proof of Taylor's Theorem: Construction

- Rewrite (1) while considering an $A \in \Re$ given by

$$h^{n+1-u}A := f(a+h) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k =: R_n(h) \tag{4}$$

- Construct a function $g(t)$ on the closed interval $[a, a+h]$:

$$g(t) = -(a+h-t)^{n+1-u}A + f(a+h) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(a+h-t)^k \tag{5}$$

- $0^0 = 1$ and $0! = 1 \implies g(a+h) = f(a+h) - f(a+h) = 0$.

- Class exercise: Show that $g(a) = 0$.

# Proof of Taylor's Theorem: Computation of $g'(t)$

- Direct computation from (5)

$$g'(t) =(n+1-u)(a+h-t)^{n-u}A - f'(t) \\ - \sum_{k=1}^{n} \frac{(a+h-t)^k}{k!} f^{(k+1)}(t) + \sum_{k=1}^{n} \frac{(a+h-t)^{k-1}}{(k-1)!} f^{(k)}(t)$$

- Note that

$$-f'(t) - \sum_{k=1}^{n} \frac{(a+h-t)^k}{k!} f^{(k+1)}(t) = -\sum_{k=1}^{n+1} \frac{(a+h-t)^{k-1}}{(k-1)!} f^{(k)}(t)$$

- Hence,
  $$g'(t) = (n+1-u)(a+h-t)^{n-u}A - \frac{(a+h-t)^n}{n!} f^{(n+1)}(t).$$

# Proof of Taylor's Theorem: Obtaining Remainder

- Since $0 = g(a) = g(a+h)$, the continuous $g$ must attain either a maximum or a minimum somewhere between $a$ and $a+h$. Hence, there exists $0 < \theta < 1$ such that

$$g'(a+\theta h) = 0.$$

- Accordingly,

$$0 = g'(a+\theta h) = (n+1-u)(h-\theta h)^{n-u} A - \frac{(h-\theta h)^n}{n!} f^{(n+1)}(a+\theta h),$$

from which we obtain

$$A = \frac{h^u (1-\theta)^u}{(n+1-u)n!} f^{(n+1)}(a+\theta h).$$

- Substituting this $A$ into (4) yields (2). ☐

# Integral Taylor's Theorem

**Theorem** Assume $f$ has a continuous second derivative $f''$ in some neighborhood of $a \in \Re$. Then, for every $x$ in this neighborhood, we have

$$f(x) = f(a) + f'(a)(x-a) + E_1(x),$$

where

$$E_1(x) = \int_a^x (x-t) f''(t) dt.$$

# Proof of Integral Taylor's Theorem

- By definition,

$$E_1(x) = f(x) - f(a) - f'(a)(x-a) = \int_a^x f'(t)dt - f'(a)\int_a^x dt$$
$$= \int_a^x (f'(t) - f'(a))dt.$$

- Let $u := f'(t) - f'(a)$. Note that $du/dt = f''(t)$.
- Let $v = t - x$, and $dv = dt$. Hence, with integration by parts,

$$E_1(x) = \int_a^x u\,dv = uv|_a^x - \int_a^x (t-x)f''(t)dt.$$

- Now, $uv|_a^x = (f'(t) - f'(a))(t-x)|_a^x = 0$, and thus $E_1(x) = \int_a^x (x-t)f''(t)dt$.

# Alternative Form of Taylor's Theorem

- Let $x_0 = a$.
- Also, for a generic variable $\epsilon \in \Re$, we express $x$ as $x_0 + \epsilon$.
- Accordingly, $x - a$ becomes $x_0 + \epsilon - x_0 = \epsilon < 1$, and Taylor's theorem (3) becomes

$$\begin{aligned} f(x_0 + \epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{f''(x_0)}{2!}\epsilon^2 \\ + \cdots + \frac{f^{(n)}(x_0)}{n!}\epsilon^n + R_n(\epsilon). \end{aligned}$$

- Since $x_0$ can be any point on $\Re$, we write it as $x$ instead. Consequently,

$$\boxed{f(x + \epsilon) = f(x) + f'(x)\epsilon + \frac{f''(x)}{2!}\epsilon^2 + \cdots + \frac{f^{(n)}(x)}{n!}\epsilon^n + R_n(\epsilon)}$$

# Concept Checkers

- Suppose the function $f(x)$ is smooth, i.e., infinitely differentiable. In (3), if $a = 0$, Taylor's theorem produces an expansion series in your Pre-U math. What is the name of the series?

- What are the first two terms when expanding $\ln(1+x)$ using the expansion series above?

- What is the alternative form for integral Taylor's theorem (Slide 11)?

- Show that $\ln x$ can be expanded as follows:

$$\ln x = \left(x - \frac{1}{x}\right) - \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) + \frac{1}{3}\left(x^3 - \frac{1}{x^3}\right) + \cdots$$