Static Replication

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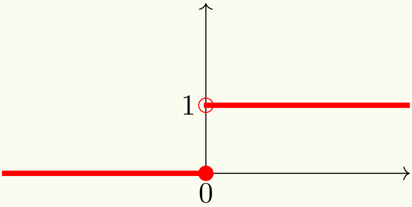
Lesson Plan

What is a Unit Step Function?

- Definition

$$1_{x>0} := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$



- Mathematical identity

$$1_{x>0} + 1_{x\leq 0} = 1.$$

Integration of Step Function

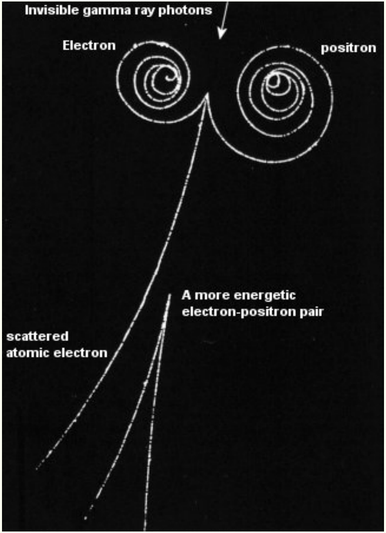
- There is nothing sacrosanct about $(0,0)$. You can shift the origin to a non-zero number a.
- Recall that $x^+ := \max(x,0)$. By shifting, we have $(x-a)^+ = \max(x-a,0)$.
- Let λ and a be positive and finite real numbers. Then

$$\int_{-\infty}^{\lambda} 1_{x>a}\,dx = (x-a)^+\Big|_0^{\lambda} = (\lambda-a)^+.$$

- On the other hand,

$$\int_{\lambda}^{\infty} 1_{x\leq a}\,dx = -(a-x)^+\Big|_{\lambda}^{\infty} = (a-\lambda)^+.$$

Anti-Electron: Positron



Picture source: Bubble Chamber

Dirac's equation

$$i\hbar\gamma^{\mu}\partial_{\mu}\psi = mc\,\psi \qquad (1)$$

Which One is Paul Adrien Maurice Dirac?

Picture source: Paul Dirac and the religion of mathematical beauty

Mathematics and Beauty

We must admit that religion is a jumble of false assertions, with no basis in reality. The very idea of God is a product of the human imagination.

— Dirac (1927), atheist

Source: Physics and Beyond : Encounters and Conversations (1971) by Werner Heisenberg, pp. 85-86

God used beautiful mathematics in creating the world.

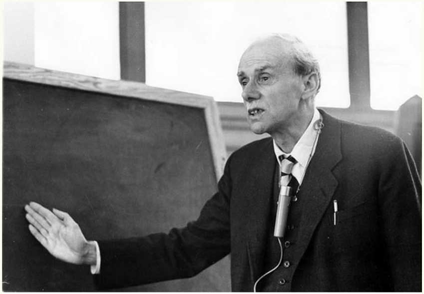
— Dirac (1963), ex-atheist

Source: The Cosmic Code: Quantum Physics As The Language Of Nature (2012) by Heinz Pagels, pp. 295

Conversion by Mathematical Beauty

"God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe."

The Evolution of the Physicist's Picture of Nature
Scientific American, May 1963



Paul A. M. Dirac (1962)
from AIP Emilio Segre Visual Archives

Source: http://ysfine.com/dirac/dirac44.jpg

Dirac's Delta "Function"

- Definition

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \delta(x)dx = 1 \\ &\delta(x) = 0 \qquad \text{for } x \neq 0. \end{aligned}\right\} \tag{2}$$

- The most important property of $\delta(x)$ is exemplified by the following equation,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0), \tag{3}$$

where $f(x)$ is any continuous function of x.

Dirac's Delta "Function" (cont'd)

- By making a shift of origin to a for Dirac's δ function, we can deduce the formula

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a). \tag{4}$$

- The process of multiplying a function of x by $\delta(x-a)$ and integrating over all x is equivalent to the process of substituting a for x.

- The range of integration need not be from $-\infty$ to ∞. Any domain, say the interval $(-g_2, g_1)$ containing the critical point at which $\delta(x)$ does not vanish, will do.

Dirac's Alternative Definition of $\delta(x)$

- Consider the differential coefficient $\epsilon'(x)$ of the **step function** $\epsilon(x)$ given by

$$1_{x>0} := \left\{ \begin{array}{ll} \epsilon(x) = 1 & \text{if } x > 0 \\ \epsilon(x) = 0 & \text{if } x \leq 0. \end{array} \right\} \quad (5)$$

- Substitute $\epsilon'(x)$ for $\delta(x)$ in the left side of (3). For positive g_1 and g_2, integration by parts leads to

$$\begin{aligned} \int_{-g_2}^{g_1} f(x)\epsilon'(x)dx &= f(x)\epsilon(x)\Big|_{-g_2}^{g_1} - \int_{-g_2}^{g_1} f'(x)\epsilon(x)dx \\ &= f(g_1) - \int_0^{g_1} f'(x)dx \\ &= f(0) \end{aligned}$$

Dirac's Delta Function in $\mathbb{Q}$F

- For any payoff function $f(S)$, the origin-shifting property of the Dirac function allows us to write, for any non-negative λ,

$$
\begin{aligned}
f(S) &= \int_0^\infty f(K)\delta(K-S)dK \\
&= \int_0^\lambda f(K)\delta(K-S)dK + \int_\lambda^\infty f(K)\delta(K-S)dK
\end{aligned}
$$

- Integrating each integral by parts results in

$$
\begin{aligned}
f(S) = f(K)1_{S<K}\Big|_0^\lambda &- \int_0^\lambda f'(K)1_{S<K}dK \\
&- f(K)1_{S\geq K}\Big|_\lambda^\infty + \int_\lambda^\infty f'(K)1_{S\geq K}dK
\end{aligned}
$$

Replication by Bonds and Options

- Integrating each integral by parts once more!

$$\begin{aligned} f(S) &= f(\lambda)1_{S<\lambda} - f'(K)(K-S)^+\Big|_0^\lambda + \int_0^\lambda f''(K)(K-S)^+ dK \\ &\quad + f(\lambda)1_{S\geq\lambda} - f'(K)(S-K)^+\Big|_\lambda^\infty + \int_\lambda^\infty f''(K)(S-K)^+ dK \\ &= f(\lambda) + f'(\lambda)\left[(S-\lambda)^+ - (\lambda-S)^+\right] \\ &\quad + \int_0^\lambda f''(K)(K-S)^+ dK + \int_\lambda^\infty f''(K)(S-K)^+ dK. \\ &= f(\lambda) + f'(\lambda)(S-\lambda) \\ &\quad + \int_0^\lambda f''(K)(K-S)^+ dK + \int_\lambda^\infty f''(K)(S-K)^+ dK. \qquad (6) \end{aligned}$$

Static Replication

- The payoff $f(S)$ contingent on the outcome S at maturity T can be replicated by
  - $f(\lambda)$: number of risk-free discount bonds, each paying \$1 at T
  - $f'(\lambda)$: number of forward contracts with delivery price λ
  - $(K-S)^+$: European put option's payoff at T of strike K
  - $(S-K)^+$: European call option's payoff at T of strike K
  - $f''(\lambda)dK$ is the number of put options of all strikes $K<\lambda$, and call options of all strikes $K>\lambda$

- The payoff replication is static, and model-free of Type 1.

Price of Static Replication

- The cost of replicating the payoff $f(S_T)$ is the price of the contract that provides the payoff.

$$\mathbb{E}_0^{\mathbb{Q}}\left(f(S_T)\right) = \mathbb{E}_0^{\mathbb{Q}}\left(f(\lambda)\right) \tag{7}$$

$$+ f'(\lambda)\left(\mathbb{E}_0^{\mathbb{Q}}(S_T) - \lambda\right) \tag{8}$$

$$+ \int_0^{\lambda} f''(K)\,\mathbb{E}_0^{\mathbb{Q}}\left((K - S_T)^+\right) dK \tag{9}$$

$$+ \int_{\lambda}^{\infty} f''(K)\,\mathbb{E}_0^{\mathbb{Q}}\left((S_T - K)^+\right) dK \tag{10}$$

- By the first principle, the cost or price is, given the risk-free interest rate r_0,

$$\text{price} = e^{-r_0 T}\,\mathbb{E}_0^{\mathbb{Q}}\left(f(S_T)\right).$$

Risk Neutral Pricing

- The first piece is the bond $f(\lambda)$, so the price is $e^{-r_0 T} f(\lambda)$.
- The pricing formulas for options according to the first principle of $\mathbb{Q}$F are

$$c_0(K) = e^{-r_0 T} \mathbb{E}_0^{\mathbb{Q}} \left[\left(S_T - K \right)^+ \right]; \quad (11)$$

$$p_0(K) = e^{-r_0 T} \mathbb{E}_0^{\mathbb{Q}} \left[\left(K - S_T \right)^+ \right]. \quad (12)$$

- The second piece is a forward but its value can be replicated by a pair of put and call struck at λ (see (6)). Hence, its price is

$$f'(\lambda)\big(c_0(\lambda) - p_0(\lambda)\big).$$

Price of Static Replication

- The total replication cost at time 0 (one contract), is

$$
\begin{aligned}
e^{-r_0 T}\,\mathbb{E}_0^{\mathbb{Q}}\big(f(S_T)\big) = e^{-r_0 T} f(\lambda) & \\
&+ f'(\lambda)\big(c_0(\lambda) - p_0(\lambda)\big) \\
&+ \int_0^{\lambda} f''(K) p_0(K) dK \\
&+ \int_{\lambda}^{\infty} f''(K) c_0(K) dK
\end{aligned}
$$

Example: Log Contract

- Let $f(x) = \log(x)$. Then $f'(x) = \dfrac{1}{x}$, and $f''(x) = -\dfrac{1}{x^2}$. We set $S = S_T$, and $\lambda = F_0$. It follows that

$$\log(S_T) = \log(F_0) + \frac{1}{F_0}(S_T - F_0) - \int_0^{F_0} \frac{(K - F_0)^+}{K^2} dK - \int_{F_0}^{\infty} \frac{(F_0 - K)^+}{K^2} dK$$

- Accordingly,

$$\log\left(\frac{S_T}{F_0}\right) = \frac{1}{F_0}(S_T - F_0) - \int_0^{F_0} \frac{(K - F_0)^+}{K^2} dK - \int_{F_0}^{\infty} \frac{(F_0 - K)^+}{K^2} dK. \quad (13)$$

Risk Neutral Expectation

- Under the risk neutral measure,

$$\mathbb{E}_0^{\mathbb{Q}}(S_T) = F_0$$

- Consequently,

$$\mathbb{E}_0^{\mathbb{Q}}\left(\frac{1}{F_0}(S_T - F_0)\right) = \frac{1}{F_0}\left[\mathbb{E}_0^{\mathbb{Q}}(S_T) - F_0\right] = 0.$$

- Therefore, under the risk-neutral measre $\mathbb{Q}$, (13) becomes

$$\mathbb{E}_0^{\mathbb{Q}}\left[\log\left(\frac{S_T}{F_0}\right)\right] = -e^{r_0 T}\int_{F_0}^{\infty}\frac{c_0(K)}{K^2}\,dK - e^{r_0 T}\int_0^{F_0}\frac{p_0(K)}{K^2}\,dK. \tag{14}$$

Partition

- A **partition** of an interval $[a, b]$ is a finite sequence of numbers of the form

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

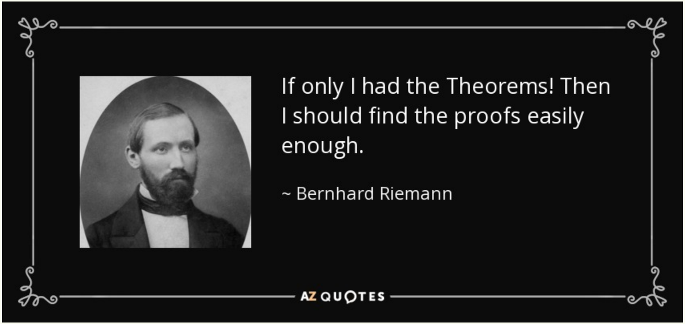
  Each $[x_i, x_{i+1}]$ is called a subinterval of the partition.

- The mesh or **norm** of a partition is defined as the length of the longest subinterval, i.e.,

$$\max\left(x_{i+1} - x_i\right), \quad i \in [0, n-1].$$

- A **tagged partition** $P(x, t)$ of an interval $[a, b]$ is a partition together with a finite sequence of numbers $t_0, \ldots, t_{n-1}$ subject to the conditions that for each i, $t_i \in [x_i, x_{i+1}]$.

Who is Riemann?



- A genius who lived for only 39 years.
- Every theoretical physicist will study Riemannian geometry.
- Every mathematician will dream to solve the **Riemannian hypothesis**.

Riemann Sums and Riemann Integral

- Let f be a real-valued function defined on the interval $[a, b]$. The Riemann sum of f with respect to the tagged partition $x_0, \ldots, x_n$ together with $t_0, \ldots, t_{n-1}$ is

$$\sum_{i=0}^{n-1} f(t_i)\,(x_{i+1} - x_i) =: \sum_{i=0}^{n-1} f(t_i)\Delta x_{i+1}.$$

- Note that $f(t_i)\Delta x_{i+1}$ is the area of a rectangle.
- The Riemann integral is the limit of the Riemann sums of a function as the partitions get finer. If the limit exists then the function is said to be **Riemann-integrable**.
- The Riemann sum can be made as close as desired to the Riemann integral by making the partition fine enough.

Discrete Replication

- Strike price is never continuous; the strike price interval is ΔK.
- The integral can be represented as a Riemann sum.
- Let K_0 be the smallest strike in the option chain. Suppose K_p is the largest strike less than λ.

$$\int_0^{\lambda} f''(K) p_0(K) dK \approx \sum_{i=1}^{p} f''(K_i) p_0(K_i) \Delta K$$

- Let K_h be the largest strike in the option chain. Suppose K_c is the smallest strike greater than λ.

$$\int_{\lambda}^{\infty} f''(K) c_0(K) dK \approx \sum_{j=c}^{h} f''(K_j) c_0(K_j) \Delta K$$

Discrete Replication (cont'd)

- If the strike price is not uniform, ΔK for strike K_i becomes

$$\frac{\Delta K_{i-1} + \Delta K_{i+1}}{2}.$$

- What about λ?
- Answer: A simple practice is

$$f''(K_\lambda)\big(c_0(K_\lambda) + p_0(K_\lambda)\big)\Delta K,$$

  where K_λ is the strike price closest to λ.
- Discrete replication is only feasible in the real world.
- Why?

Mathematical Beauty

" ... it turned out that the equations that really work in describing nature with the most generality and the greatest simplicity are very elegant and subtle. It's the kind of beauty that might be hard to explain [but] is just as real to anyone who's experienced it as the beauty of music."

Source: **Viewpoints on String Theory**

Source: **Edward Witten's 2014 Kyoto Prize Commemorative Lecture in Basic Sciences**

Key Points

- Dirac's delta "function" is an impulse function.
- Pre-U's integration by part + brilliant ideas lead to an elegant replication strategy.
- In $\mathbb{Q}$F, cost of replication is the fair price of the payoff under the risk-neutral measure $\mathbb{Q}$.
- Sense of beauty
  - ★ Quants apply mathematics to create beautifully useful models.
  - ★ Quants solve models' limitations with cleaver approximation and implementation to make them beautifully useful.

The greatest strategy is doomed if it's implemented badly.

Bernhard Riemann

Source: AZ Quotes

Assignment: Replication Cost

Call			Put	
Bid	Ask	Strike	Bid	Ask
137.7	141.6	5,840	54.2	57.4
130.4	134.2	5,850	56.9	60.1
123.6	126.7	5,860	59.7	62.9
116.5	119.7	5,870	62.6	65.9
113.1	116.3	5,875	64.2	67.4
109.4	112.9	5,880	65.7	69.0
102.7	106.1	5,890	68.9	72.3
96.2	99.6	5,900	72.4	75.7
89.8	93.2	5,910	76.0	79.4
83.7	**86.9**	**5,920**	**79.7**	**83.2**
80.6	83.9	5,925	81.6	85.2
77.7	80.9	5,930	83.6	87.2
71.8	75	5,940	87.7	91.4
66.2	69.4	5,950	92.1	95.7
60.8	63.9	5,960	96.6	100.4
55.7	58.7	5,970	101.4	105.2
53.1	56.2	5,975	103.8	107.7
50.7	53.7	5,980	106.3	110.2
46.0	49.0	5,990	111.5	115.5

- Let $\lambda = F_0$ = 5920, $r_0 = 1\%$. and T = 29/365.
- The payoff function is
  $$f(x) = (x + 5919)(x - 5919)$$
- You have to buy at the ask price and sell a the bid price.
- What is the cost of the bond?
- What is the cost of the forward?
- What is the cost of the integral involving call options?
- What is the cost of the integral involving put options?