

Principles of Quantitative Finance

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Uncertainty Quotes by Famous Scientists

-  “As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.”

-  “Il n'est pas certain que tout soit incertain.”

Uncertainty



What is uncertainty?

Uncertainty arises when certainty becomes probabilistic.



Almost 100% chance (**known knowns**):

- What is the probability of the sun rising tomorrow?
- What is the probability of a human being who will not die?



Never 100% certain (**known unknowns**):

- What is the probability for OCBC to lose 1 million dollars tomorrow?
- What is the probability of OCBC's stock going down (negative return) tomorrow?



In the probabilistic framework, uncertainty is about events that are known unknowns.



The notion of “unknown knowns” is an oxymoron. But what about “**unknown unknowns**”? Is there such thing?

Risk



For the two examples of known unknowns, which of it could be a risk to OCBC's stock holder?

Answer:_____



Is uncertainty a necessary condition of risk?

Answer:_____



Is uncertainty a sufficient condition of risk?

Answer:_____



What drives investors and traders to take risk?

An Example of Risk



Suppose an investor buys 1,000 shares of a stock at 5 dollars per share and holds the shares for a year. Suppose the share price turns out to be 6 dollars a year later. Moreover, if the stock does not pay dividends, then the **realized return** after selling the shares will be

$$\frac{6 - 5}{5} = 20\%.$$



Having pocketed the **gross profit** of \$1,000, the **capital** gain of 20% is the return realized over a year.

An Example of Risk (cont'd)



If the round-trip transaction fees in the form of brokerage commission, clearing and settlement, and any other costs amount to \$0.2 per share, the return net of these costs will be less than 20%:

$$\frac{6 - 5 - 0.2}{5} = 16\%.$$

The profit after fees is \$800, and the resulting 16% is referred to as the **net return**.



Moreover, if the investor has to pay tax on the capital gain, the net return after tax is even smaller, say, 12%.



The bottom line is, stock return is uncertain, and the downside aspect of this uncertainty is the risk that the investor takes.

An Example of Risk (cont'd)



Uncertainty that may trigger financial losses is risk.



In the earlier example, if the stock price declines to \$4 a year later, then the net return will become

$$\frac{4 - 5 - 0.2}{5} = -24\%.$$

Accordingly, the investor losses 24%, i.e., every \$1.00 the investor had a year earlier is reduced to \$0.76.



But a **stock analyst** who does not invest in the stock will have no risk, though the analyst recognizes that the stock return is uncertain and risky too.

Risk-Free Investment



The alternative to investing in stock is to deposit the money at a **totally trustworthy bank**.



In this case, there is no uncertainty whatsoever and hence risk-free. This is because the depositor is guaranteed to receive the money deposited plus interest **at the point of time when fund is deposited**.



The realized return, which is the interest rate quoted by the bank, is known in advance.



By contrast, the realized return on a stock can only be known a year later in the example above.

Risk-Free Asset and Risk-Free Rate






In general, a **risk-free asset** is a financial instrument for which the future cash flow is guaranteed and exactly predictable today.



There is absolutely no uncertainty whatsoever, and the value of such asset increases at a constant **risk-free rate** r_0 .

First Principle of QF

-  The **first principle** of Quantitative Finance states that a risk-free asset whose value is zero remains zero, or when non-zero, increases at a constant **risk-free rate**.
-  It is known more widely as the **time value of money**, i.e., a dollar received today is more valuable than a dollar received in the future. Cash is king.
-  **Any portfolio that is risk-free must earn only risk-free interest rate r_0 .**

Forward and Backward Values

✂ Denote the cash flow as C and T is the maturity of lending and borrowing.

✂ The **forward value** (FV) and the **present value** (PV) are given by, respectively,

$$\text{FV} = C(1 + r_0)^T \quad (1)$$

$$\text{PV} = \frac{C}{(1 + r_0)^T} \quad (2)$$

✂ Under the risk-free framework, the forward value and the present value are both sides of the same coin.

$$\text{PV} \xleftrightarrow{r_0} \text{FV}.$$

Tutorial: Payoff Function

✂ Payoff is the cash flow at maturity before costs, taxes, etc.

✂ What is the payoff of investing in a risk-free security by lending \$10,000 at 2% with maturity in two years' time?

Answer: _____

✂ Plot this risk-free payoff against all the possible values of OCBC's stock price S_T at time T .

✂ What do you call the function plotted?

Answer: _____

✂ In general, the payoff is a function $f(S_T)$.

First Principle of $\mathbb{Q}F$ in General



When the cash flow in (2) is the **expected value** of some **payoff function** in the future time T instead, the present value at time 0 is

$$\text{PV} = \frac{\mathbb{E}^{\mathbb{Q}}(\text{payoff function})}{(1 + r_0)^T}. \quad (3)$$

The symbol $\mathbb{E}^{\mathbb{Q}}(\cdot)$ indicates the expectation of the random variable within the brackets.



Condition: The functional form of the **payoff function** $f(\mathbf{X}_T)$ must be known, and the investment horizon or expiration date T must be known at time 0.

Risk-Neutral

- ✂ When discounted by the risk-free rate, the expectation is said to be taken under the **risk-neutral** probability measure denoted by \mathbb{Q} .
- ✂ It is risk-neutral in the sense that investors are **indifferent to risk** because the payoff function can be **replicated** exactly, at least in principle, by a combination of other financial instruments.
- ✂ As a result of this replication possibility, investors can lay off the risk associated with the payoff of a financial instrument by an **off-setting** or **neutralizing replication** and thus becoming neutral to the payoff.

Tutorial: Replication

- ✂ Suppose the world is binary: the stock price either rises to \$6 or drops to \$4 from the current price of \$5 ($= S_0$).
- ✂ Suppose a bank structures a financial contract whose payoff depends on the outcome of stock price S_T at time T :

$$\text{Payoff} = \begin{cases} \$70, & \text{if stock price goes up to \$6 per share;} \\ \$50, & \text{if stock price goes down to \$4 per share.} \end{cases}$$

- ✂ Suppose the risk-free rate is 1% and T is 1 year.

Tutorial: Replication (cont'd)



This payoff can be replicated by a portfolio of x shares and y dollar amount of the risk-free asset to cover the two possibilities. Replication requires the portfolio's dollar amount to equal the payoff:

$$x\$6 + \$y(1 + 0.01) = \$70$$

$$x\$4 + \$y(1 + 0.01) = \$50$$



So we obtain a system of two linear equations with two unknowns, x and y . Using Pre-U math, you can solve it easily and obtain

$$x = 10 \text{ shares}, \quad y = \$9.90$$

Tutorial: Replication (cont'd)

- ✂ Accordingly, by having 10 shares and lending out \$9.90 to the 100% trustworthy financial institution, the bank is neutral to whether the stock price goes up or down.
- ✂ Notice that you don't have to know the probability of stock going up or down in the replication!
- ✂ The bank sells this “structured product” to the client who wants this payoff. The bank is indifferent to the outcome of stock movement because the payoff can be replicated.
- ✂ How much should the bank sell this product? (Hint: What's the **cost of replication**?)

Answer: _____

Risk and Return

- Higher risk, higher return?
- The **second principle** of Quantitative Finance states that the return on a risky asset is **expected** or **required** to be at a higher rate than that of a risk-free asset. Moreover, when the risk is higher, the **expected return** must be higher too. We write

$$\mathbb{E}(r_a) \geq \mathbb{E}(r_b), \quad \text{if asset } a\text{'s risk } \geq \text{asset } b\text{'s risk.}$$

Here, r_a and r_b are, respectively, the return on asset a and on asset b .

Risk Premium

- The first principle provides the baseline framework to examine risk.
- The second principle suggests how risk alters the expected rate of return required by investors.
- Suppose the expected return on a stock is 10% per year and the risk-free rate is 3%. The spread or the difference of 7% between the risky asset and the risk-free asset is the **risk premium**. It compensates investors for exposing themselves to risk. Generically, we write

$$\text{risk premium} = \mathbb{E}(r_a) - r_0. \quad (4)$$

Market Risk Premium

- Expected return is just a concept.
- Models are needed to make expected return tractable for analysis.
- A popular type of risky asset is the **market portfolio**. It comprises all tradable stocks.
- Equation (4) suggests that the expected return on the market portfolio denoted by $\mathbb{E}(r_m)$ less the risk-free rate r_0 is known as the **market risk premium**:

$$\text{market risk premium} = \mathbb{E}(r_m) - r_0. \quad (5)$$

Example 1: Capital Asset Pricing Model

- ☀️ CAPM suggests that the risk premium of a stock is directly proportional to the market risk premium.

$$\mathbb{E}(r_a) - r_0 = \beta_a \left(\mathbb{E}(r_m) - r_0 \right). \quad (6)$$

- ☀️ The proportional constant β_a is interpretable as the sensitivity of portfolio a 's risk premium to the market portfolio's risk premium.

- ☀️ **Implication**

A smaller beta has lesser risk compared to a larger beta.

Accordingly, for stock a and stock b with positive β_a and β_b , respectively, under CAPM,

$$\mathbb{E}(r_a) \geq \mathbb{E}(r_b), \quad \text{if } \beta_a \geq \beta_b.$$

Example 2: Modigliani-Miller Proposition II

- ✱ In this model, a company may decide to be completely free of debt (**unlevered**) or to take on a debt for **leverage** (**levered**).
- ✱ In the **unlevered** scenario, the rate of return on the equity E required by the shareholder is $\mathbb{E}(r_u)$, which becomes $\mathbb{E}(r_l)$ when the company is levered with debt D .
- ✱ The creditor evaluates the company's financial health and demands the rate of return $\mathbb{E}(r_d)$ for lending money to the firm. The risk-return proposition of Modigliani and Miller is remarkably similar to CAPM (6):

$$\mathbb{E}(r_l) - \mathbb{E}(r_u) = \gamma \left(\mathbb{E}(r_u) - \mathbb{E}(r_d) \right). \quad (7)$$

Here, γ is the **debt-to-equity ratio** D/E .

Consenting Traders



The **third principle** of Quantitative Finance states that in the financial market, for every willing buyer who pays a price for an asset, there is a willing seller who sells that asset at the same price.

Louis Bachelier

Il semble que le marché c'est-à-dire l'ensemble des spéculateurs, ne doit croire à *un instant donné* ni à la hausse, ni à la baisse, puisque, pour chaque cours coté, il y a autant d'acheteurs que de vendeurs.

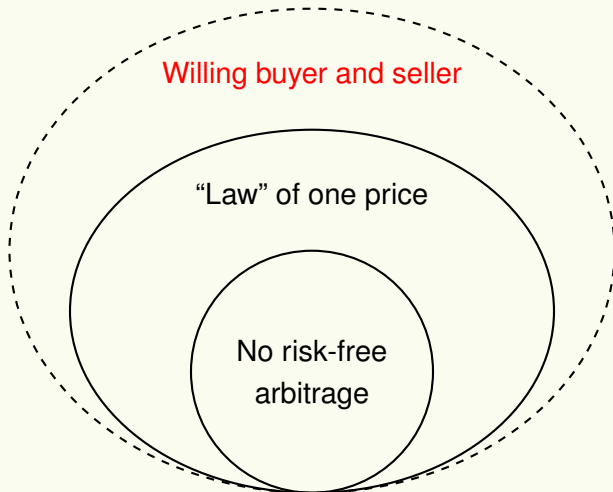


Biography

Théorie de la Spéculation (1900)

It seems that the market, i.e. the ensemble of speculators, must believe *at a given instant* neither a rise, nor a fall, since, for every quoted price, there are as many buyers as sellers.

Third Principle Illustrated



No risk-free arbitrage opportunity is the most restrictive form of the third principle.

Relative Valuation



For any two portfolios A and B generating, respectively, future values or cash flows of $V_A(T)$ and $V_B(T)$ that are **relatively comparable** at current time t ($t < T$), i.e.,

$$V_A(T) \leq V_B(T).$$



then the third principle of Quantitative Finance suggests that their current prices $P_A(t)$ and $P_B(t)$ must also abide by the same relative relationship, i.e.,

$$P_A(t) \leq P_B(t).$$



Otherwise, whenever $P_A(t) > P_B(t)$, by the third principle of QF, the **arbitrageurs** will step in to sell asset A and buy asset B at time t , and unwind this long-short position at time T to capture the free money.

Principles Versus Models



Being the basis for reasoning, the three **principles**, however, do not provide details on how the expected return and risk should be defined and measured.



The details are provided by **models** instead. For example, CAPM is a model for expected returns and risks. It is a popular model used by the practitioners to price the **market risk**.

What's a Model?

A model is a metaphor of limited applicability, not the thing itself.

Emanuel Derman



To Derman, a model is no more than “a caricature that overemphasizes some features at the expense of others.”



In this context, CAPM is quite limited, for there are other types of risks, such as credit risk, liquidity risk, operational risk, and even **model risk**, which are not addressed by CAPM.

Model Risk



Risk that models are not handled properly, either they are inapplicable, implemented incorrectly, or applied on data & calibration issues.



Shooting game in an amusement park



Check key assumptions, model boundaries, “my baby” syndrome, numerical schemes, data inputs, etc.



How to model model risk?

Newton's Three Laws of Motion



First Law

Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.



Second Law




The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.



Third Law

To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Principles of QF Versus Newton's Laws of Motion

-  The first principle of QF states categorically that there is a **risk-free frame of reference**, in which the value of a risk-free asset increases by the constant yield till maturity.
-  According to the second principle of QF, the value of a risky asset in the future is expected to increase at a higher rate than that of a risk-free asset. Moreover, when the risk is higher, the expected return is higher too.
-  According to the third principle of QF, for every buyer who pays a price for an asset, there is a seller who sells that asset at the same price. It follows that two assets of the same expected return and risk will have the same present value. Otherwise, there will be a risk-free arbitrage opportunity.

Physics Envy

Physics

Thermodynamic equilibrium

Inertial frame of reference

Constant velocity

Forces

Inverse of mass

Equal action and reaction

No perpetual motion

Friction

Quantitative Finance

Supply equals demand

Risk-free frame of reference

Constant risk-free rate

Risk factors/premiums

CAPM's beta (a model)

As many buyers as sellers

No risk-free arbitrage

Market friction

Fundamental Chasm

- ☎ Laws of physics are dealing with real objects that cannot will.
- ☎ The three principles are distillations of human behaviors. Humans should follow the principles but they can be irrational at times.

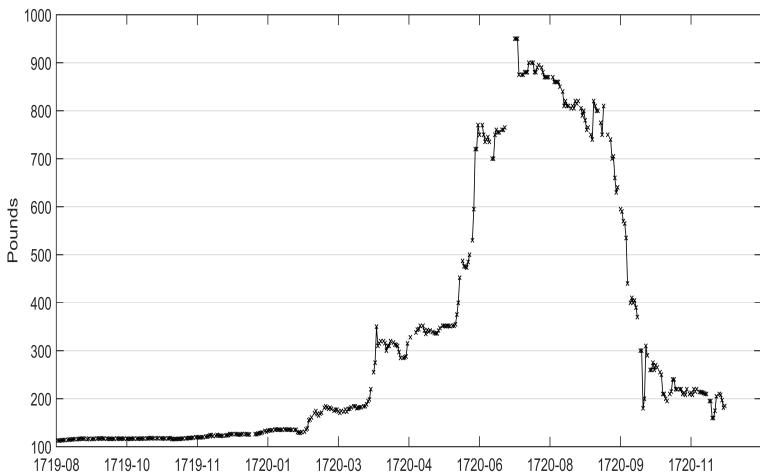
What really matters is an awareness of how greed and fear can drive rational people to behave in strange ways when they gather in the marketplace.

Joseph de la Vega



[Wikipedia](#)

Newton's Loss



Stock prices of South Sea Company in British pounds

Source: Frehen, Goetzmann, and Rouwenhorst (2012)

Madness

“I can calculate the motions of erratic bodies, but not the madness of a multitude.”



Picture source: [MacTutor History of Mathematics archive](#)

Takeaways

- 👉 No uncertainty, no risk.
- 👉 Risk-neutral measure is important in pricing.
- 👉 Replication of payoff allows you to price a derivative in practice.
- 👉 The three principles of Quantitative Finance are useful guides.
- 👉 Principles are not models.
- 👉 Using any model incurs a model risk.

Assignment from Chapter 3

Exercises

Questions 1 to 3 of the book

Week 3 Additional Exercises

- 1 Having derived the price (PV) of the structured product in the replication tutorial (see the three slides from Slide 16), you can back out the probability of the stock price going up in this binary economy. So what is the probability of $S_1 = \$6$?
- 2 Everything being the same as in the replication tutorial except the payoff:

$$\text{Payoff} = \begin{cases} \$70, & \text{if stock price goes up to } \$6 \text{ per share;} \\ -\$70, & \text{if stock price goes down to } \$4 \text{ per share.} \end{cases}$$

- a. Determine the number of shares x and the amount of bond y .
 - b. Compute the present value of the structured product.
 - c. What is the probability of $S_1 = \$4$?
- 3 Provide a proof for the relative valuation in Slide 27.