

L'Hôpital's Rule

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L'Hôpital's Rule



Suppose that f and g are continuously differentiable at a real number c , and that $f(c) = g(c) = 0$.





Then

$$\begin{aligned}\lim_{x \rightarrow c} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow c} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{g(x) - g(c)} \\ &= \lim_{x \rightarrow c} \frac{\left(\frac{f(x) - f(c)}{x - c} \right)}{\left(\frac{g(x) - g(c)}{x - c} \right)} = \frac{\lim_{x \rightarrow c} \left(\frac{f(x) - f(c)}{x - c} \right)}{\lim_{x \rightarrow c} \left(\frac{g(x) - g(c)}{x - c} \right)} \\ &= \frac{f'(c)}{g'(c)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.\end{aligned}$$


Source: [Wikipedia](#)

Important Remarks

 If $f'(c) = g'(c) = 0$, we can apply L'Hôpital's rule again.

 A general proof suggests that if $\lim_{x \rightarrow c} f(x) = \pm\infty$ and $\lim_{x \rightarrow c} g(x) = \pm\infty$. Then we also have

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}.$$

 In other words, L'Hôpital's rule is applicable for indeterminate form of either $\frac{0}{0}$ type or $\frac{\pm\infty}{\pm\infty}$ type.

 L'Hôpital's rule is still applicable even when the limit is $\pm\infty$.

Examples



$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$$

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty.$$



$$\lim_{x \downarrow 0} x \log x$$

$$\lim_{x \downarrow 0} x \log x = \lim_{x \downarrow 0} \frac{\log x}{\frac{1}{x}} = \lim_{x \downarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x \downarrow 0} x = 0.$$

Exercises from Marta Hidegkuti

$$1 \quad \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x^2}$$

$$2 \quad \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{5x}$$

$$3 \quad \lim_{a \rightarrow 1} \frac{3a^2 - 2a - 1}{5a^2 - a - 4}$$

$$4 \quad \lim_{y \rightarrow \infty} \frac{\ln y}{\sqrt[3]{y}}$$

$$5 \quad \lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$$

$$6 \quad \lim_{m \rightarrow 2} \frac{m^5 - 32}{m^3 - 8}$$

$$7 \quad \lim_{\theta \rightarrow \pi/2} \frac{\tan \theta}{\tan 5\theta}$$

$$8 \quad \lim_{x \rightarrow \infty} \frac{x}{\ln(x+1)}$$

$$9 \quad \lim_{x \rightarrow 0} \frac{x^3}{\tan x - x}$$

$$10 \quad \lim_{\beta \rightarrow 0} \frac{\sin \beta - \beta}{\tan \beta - \beta}$$

$$11 \quad \lim_{x \rightarrow 0} \left(x^2 e^{1/x^2} \right)$$

$$12 \quad \lim_{p \rightarrow 0} \frac{e^{3p} - 1}{\sin 2p}$$

$$13 \quad \lim_{x \rightarrow 0} \frac{e^{(x^2)} + 10}{1 - \cos x}$$

$$14 \quad \lim_{x \rightarrow 1} \frac{x^{2/3} - x^{1/2}}{x - 1}$$

$$15 \quad \lim_{\alpha \rightarrow 0} \frac{\alpha}{\arctan 2\alpha}$$