Lesson 12 Tutorial Questions

12.1 Let $C([-e,e])$ be the linear space of all real-valued functions that are continuous over the interval $[-e,e]$. What is the value of the inner product of $f(x) = x$ and $g(x) = \ln(x)$?
(Hint: You need to perform integration by parts.)

12.2 Suppose $\mathscr{M}_2(\mathfrak{R})$ denotes the set of all 2-dimensional real-valued square matrices. The inner product can also be defined for square matrices $\boldsymbol{A}$ and $\boldsymbol{B} \in \mathscr{M}_2(\mathfrak{R})$ by the trace as follows:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle := \mathrm{Tr}\big(\boldsymbol{A}'\boldsymbol{B}\big)$$

(a) Show that given this definition of inner product, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \langle \boldsymbol{B}, \boldsymbol{A} \rangle$.

(b) Show that $\langle \boldsymbol{A}+\boldsymbol{B}, \boldsymbol{C} \rangle = \langle \boldsymbol{A}, \boldsymbol{C} \rangle + \langle \boldsymbol{B}, \boldsymbol{C} \rangle$, where $\boldsymbol{C} \in \mathscr{M}_2(\mathfrak{R})$.

(c) Show that given a scalar s, $\langle s\boldsymbol{A}, \boldsymbol{B} \rangle = s\langle \boldsymbol{A}, \boldsymbol{B} \rangle$.

(d) Show that if $\boldsymbol{A} \neq \boldsymbol{0}_2$, then $\langle \boldsymbol{A}, \boldsymbol{A} \rangle \geqq 0$.
Therefore, $\mathscr{M}_2(\mathfrak{R})$ is an inner product space.

(e) Let $\boldsymbol{U} = \begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} \alpha^2 & \alpha - 1 \\ \alpha + 1 & -1 \end{bmatrix}$. Find all values of α such that $\boldsymbol{U} \perp \boldsymbol{V}$ in the inner product space $\mathscr{M}_2(\mathfrak{R})$.

12.3 Consider the following vectors:

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \boldsymbol{w}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Compute the inner product of $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$

(b) Find the orthogonal projection of the vector $\boldsymbol{w}_1$ onto $\boldsymbol{w}_2$.

(c) Using the Gram-Schmidt process, construct the orthonormal basis.

12.4 Suppose the subspace of $\mathfrak{R}^4$ is W and it is generated by two vectors:

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Find the orthogonal complement space $W^{\perp}$.