Lesson 8 Tutorial Questions

8.1 Let V be the set of all real numbers. Define an operation of "addition" by

$$\boldsymbol{x} \boxplus \boldsymbol{y} = \text{ the maximum of } \boldsymbol{x} \text{ and } \boldsymbol{y}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$. Define an operation of "scalar multiplication" by

$$\alpha \boxdot \boldsymbol{x} = \alpha \boldsymbol{x}$$

for all $\alpha \in \mathfrak{R}$ and $\boldsymbol{x} \in V$. Under the operations $\boxplus$ and $\boxdot$, the set V is not a linear space. What are the linear space axioms or laws that *fail* to hold?
(Hint: Recall the 8 laws in the definition of a linear space.)

8.2 Determine whether $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathfrak{R}^3 \;\middle|\; x_1 - 4x_2 + 5x_3 = 2 \right\}$ is a subspace of $\mathfrak{R}^3$.

8.3 Express the vector $\boldsymbol{b} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$ as a linear combination of the following vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

8.4 Let V be the vector space over $\mathfrak{R}$ of all real-valued function on the interval $[0,1]$ and let

$$W = \{ f(x) \in V \mid f(x) = f(1-x) \text{ for } x \in [0,1] \}$$

be a subset of V. Determine whether the subset W is a subspace of the vector space V.

8.5 Let M be the plane $x+y+z=0$ and N be the line $x=y=z$ in $\mathfrak{R}^3$. The purpose of this exercise is to show that $\mathfrak{R}^3 = M+N$. Three things need to be established: (i) M and N are subspaces of $\mathfrak{R}^3$ (which is *very* easy and which we omit); (ii) $\mathfrak{R}^3 = M+N$; and (iii) $M \cap N = \{\mathbf{0}\}$.

(a) To show that $\mathfrak{R}^3 = M+N$ we need $\mathfrak{R}^3 \subseteq M+N$ and $M+N \subseteq \mathfrak{R}^3$. Since $M \subseteq \mathfrak{R}^3$ and $N \subseteq \mathfrak{R}^3$, it is clear that $M+N \subseteq \mathfrak{R}^3$. So all that is required is to show that $\mathfrak{R}^3 \subseteq M+N$. That is, given a vector $\boldsymbol{x}$ in $\mathfrak{R}^3$, we must find vectors $\boldsymbol{m}$ in M and $\boldsymbol{n}$ in N such that $\boldsymbol{x} = \boldsymbol{m} + \boldsymbol{n}$. Find two such vectors.

(b) The last thing to verify is that $M \cap N = \{\mathbf{0}\}$, that is, that the only vector M and N have in common is the zero vector. Suppose that a vector $\boldsymbol{x}$ belongs to both M and N. Since $\boldsymbol{x} \in M$ it must satisfy the equation

$$x_1 + x_2 + x_3 = 0. \tag{1}$$

since $\boldsymbol{x} \in N$ it must satisfy the equations

$$x_1 = x_2 \qquad \text{and} \tag{2}$$
$$x_2 = x_3. \tag{3}$$

Solve the system of equations (1)–(3).