Lesson 4 Tutorial Questions

4.1 Suppose that L_1 and L_2 are lines in the plane. The x-intercepts of L_1 and L_2 are 5 and -1, respectively, and the respective y-intercepts are 5 and 1. At what point do the lines L_1 and L_2 intersect?

4.2 Consider the following system of equations:

$$\begin{cases} x+\ \ y+\ \ z=2 \\ x+3y+3z=0 \\ x+3y+6z=3 \end{cases}. \qquad (*)$$

Communicate your **working** clearly when you are answering the following questions:

(a) Use Gaussian elimination to put the augmented coefficient matrix into the row echelon form. The result will be $\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right]$. What are a, b, and c?

(b) Use Gauss-Jordan reduction to put the augmented coefficient matrix into the reduced row echelon form. The result will be $\left[\begin{array}{ccc|c} 1 & 0 & 0 & d \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & f \end{array}\right]$. What are d, e, and f?

(c) Solve the system of linear equations $(*)$ and present your answer as follows: Express your solutions as x = _____ , y = _____ , and z = _____ .

4.3 Consider the system of linear equations

$$\begin{cases} x+ky=1, \\ kx+y=1. \end{cases}$$

(a) Find the values of k for which the system of linear equations has
   (i) no solution.
   (ii) exactly one solution.
   (iii) infinitely many solutions.

(b) When there is exactly one solution, what is it?

4.4 Suppose your are to submit a very important homework report that involves two types of questions. Altogether there are x Type 1 questions and y Type 2 questions. The grading system is a bit complicated, but you know that

$$\begin{cases} 2x+3y \leqq 90 \\ 3x+2y \leqq 120 \end{cases}.$$

In other words, 90 is the maximum possible point under Criterion A if all the x and y questions are answered correctly. Likewise, 120 is the maximum possible point under Criterion B for giving the correct answers to all the questions. The final outcome of your report, however, is awarded according to $f(x,y) = 7x + 5y$ instead, and the maximum point is undisclosed.

Being a serious student, you want to know how many questions (x and y) there are. You also want to know the maximum point for this report, so that you can evaluate how much you need to prepare to score, say 250 points. Can you figure out x, y, and the maximum value for the function $f(x,y)$?

To help you tackle this problem, **simplex algorithm** is proposed. The simplex algorithm is a method to obtain the maximum value of a linear function $f(x,y)$ subject to a system of linear constraints.

Our task is to maximize the objective function $f(x,y) = 7x+5y$.

The simplex algorithm begins by converting the constraints and the objective function $f(x,y)$ into a system of linear equations. This conversion is done by introducing new variables called slack variables. The slack variables represent the positive difference, or slack, between the left side of an inequality and the right side of that inequality. With slack variable s_1, the inequality $2x+3y \leqq 90$ becomes

$$2x+3y+s_1 = 90.$$

Likewise, with s_2, the inequality $3x+2y \leqq 120$ becomes

$$3x+2y+s_2 = 120.$$

In addition to the slack variables, a variable z is introduced to represent the value of the objective function, which gives rise to the equation:

$$z-7x-5y = 0.$$

These three equations constitute the system of linear equations as follows:

$$\begin{cases} z - 7x - 5y = 0, & (0) \\ 2x + 3y + s_1 = 90, & (1) \\ 3x + 2y + s_2 = 120. & (2) \end{cases}$$

(a) What is the augmented matrix for this system of linear equations?
Hint:

$$\left[\begin{array}{ccccc|c} 1 & -7 & -5 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{array}\right]. \qquad \begin{array}{c}(0)\\(1)\\(2)\end{array}$$

(b) The worst case scenario is that not even a single question is answered correctly, i.e., $x=0$, $y=0$, which implies $z=0$. To satisfy the equations, the slack variables are, respectively, $s_1 = 90$ and $s_2 = 120$. Conversely, the maximum point is achieved when $s_1 = s_2 = 0$. So your task now is to perform row operations on the augmented matrix such that it becomes

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & * & * & \gamma \\ 0 & 0 & \alpha & * & * & \delta \\ 0 & \beta & 0 & * & * & \varepsilon \end{array}\right]. \qquad \begin{array}{c}(0)\\(1)\\(2)\end{array}$$

Note that * can be any number but α, β, γ, δ, and ε must be positive.

Having arrived at this form, you can figure out x, y, and the maximum point of the report. So, what are the values of x, y, and $f(x,y)$?