Suggested **S-Graded** Tutorial Solutions

1.1 The answer is (d).

The term "regular" is a direct translation of 正規 in Japanese. But in most of the English textbooks, a matrix that has an inverse is called "invertible" instead.

1.2 The answer is (b).

Any 2×2 scalar matrix $\boldsymbol{M}$ can be written as $\boldsymbol{M} = c\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, where $c \neq 0$ is a constant value. Clearly, the inverse of $\boldsymbol{M}$ is

$$\boxed{\boldsymbol{M}^{-1} = \frac{1}{c}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}},$$

which is also a scalar matrix.

1.3 (i) No. The addition $\boldsymbol{A}+\boldsymbol{B}$ is not possible because the dimension of $\boldsymbol{A}$ and that of $\boldsymbol{B}$ are different. Matrix addition is defined only for two matrices of the same dimension.

(ii) The transpose of $\boldsymbol{B}$ is $\begin{bmatrix}1 & 3 & 0 & 4\\2 & -1 & -2 & 1\end{bmatrix}$.

(iii) No. The product $\boldsymbol{C}'\boldsymbol{A}$ in not possible under matrix multiplication. The dimension of $\boldsymbol{C}'$ is 4×2 and the dimension of $\boldsymbol{A}$ is 4×4. Since the number of columns of $\boldsymbol{C}'$, which is 2, and the number of rows of $\boldsymbol{A}$ are different, matrix multiplication is impossible.

(iv)

$$\boldsymbol{a}_4 + \boldsymbol{b}_1 = \begin{bmatrix}2\\-1\\3\\2\end{bmatrix} + \begin{bmatrix}1\\3\\0\\4\end{bmatrix} = \begin{bmatrix}3\\2\\3\\6\end{bmatrix}.$$

(v) First, we compute the product $\boldsymbol{CB}$ as follows:

$$\boldsymbol{D} := \boldsymbol{CB} = \begin{bmatrix}3 & -2 & 0 & 5\\1 & 0 & -3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\3 & -1\\0 & -2\\4 & 1\end{bmatrix} = \begin{bmatrix}3-6+0+20 & 6+2+0+5\\1+0+0+16 & 2+0+6+4\end{bmatrix}$$

$$= \begin{bmatrix}17 & 13\\17 & 12\end{bmatrix}.$$

Therefore, the trace of matrix $\boldsymbol{D}$ is $\mathrm{Tr}(\boldsymbol{D}) = 17+12 = 29$.

1.4 (i) Respectivelt, we use A, B, C, and D as the abbreviations for Amy, Ben, Charles, and Dan. Also, the abbreviations for English, French, German, Hebrew, and Italian are, respectively, E, F, G, H, and I. Based on the specification, the $\boldsymbol{A}$ matrix is

$$\boldsymbol{A} = \begin{bmatrix}0 & 1 & 1 & 0 & 0\\1 & 1 & 0 & 1 & 0\\1 & 0 & 0 & 1 & 1\\1 & 0 & 1 & 1 & 1\end{bmatrix}.$$

Note: An advanced latex technique allows you to express more clearly as

$$\boldsymbol{A} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{ccccc} E & F & G & H & I \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

(ii) $\boldsymbol{A}\boldsymbol{A}' = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 2 \\ 0 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} =: \boldsymbol{B}.$

The 4-dimensional matrix $\boldsymbol{B}$ is symmetric. In alphabetical order, every row corresponds to one of the 4 persons; likewise for each column. The diagonal elements are the numbers of languages spoken by these 4 persons. For example, $b_{22} = 3$, which is interpreted as the fact that Ben speaks 3 languages. Now, for non-diagonal elements such as $b_{12} = 1$, it means the number of languages Amy can speak to Ben, which is in French, as the other two persons speak neither French nor German. Given that $b_{13} = b_{31} = 0$, it suggests that Amy and Dan do not share a common language. As another example, for $b_{34} = b_{43} = 3$, the interpretation is that Dan and Charles can speak to each other in 3 languages (English, Hebrew, and Italian).

Hence, the interpretation of $\boldsymbol{B}$ is that each entry represents the number of languages that person i can communicate in with person j. When $i = j$, it means the number of languages person i is able to speak.

1.5 We apply the formula for a 2-dimensional square matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

With $a = 7, b = 4, c = 2$, and $d = 1$, we plug these numbers into the formula to obtain

$$\boldsymbol{A}^{-1} = \frac{1}{7-8} \begin{bmatrix} 1 & -4 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -7 \end{bmatrix}.$$