Lesson 9
Linear Map

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Approach

You cannot teach a man anything, you can only help him find it within himself.

—**Galileo Galilei**

Quotable Quote

I never teach my pupils. I only attempt to provide the conditions in which they can learn.

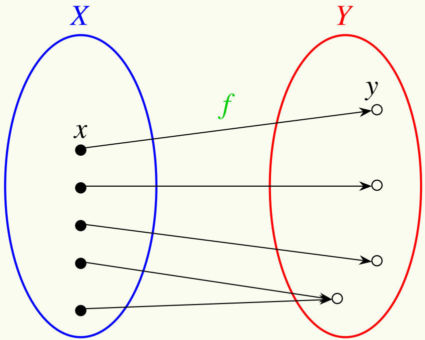
—Albert Einstein

Learning Outcomes

- Describe and compare different types of maps: onto (surjective), one-to-one (injective), bijective, composite, identity, inverse, and isomorphic maps.
- Identify the characteristics of a linear map and examine its relationship with vectors and matrices.
- Evaluate whether a transformation is linear or prove that a transformation is linear.
- Interpret the concept of representation matrix of a map.
- Describe the relationships between image, kernel, dimension of the linear space, the rank of a matrix, and how the rank is connected to injection and surjection.
- Appraise and connect the concept of isomorphic map to basis transformation and the regularity of a transformation matrix.

Map

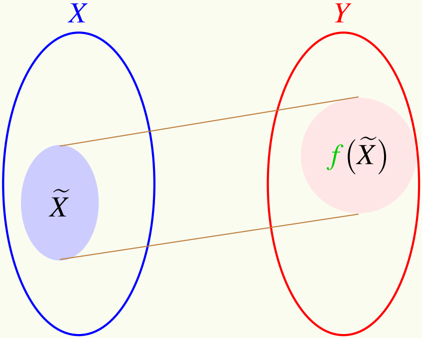
- Mapping or **map** $f : X \longrightarrow Y$.



- Example: ISOweekday $X = \{$Sunday, Monday, $\ldots$, Saturday$\}$, and $Y = \{0, 1, \ldots, 6\}$.

Mapping of Subset and Image

- The **image** $f\left(\widetilde{X}\right) = \left\{ f(x) \middle| x \in \widetilde{X} \right\}$.



- Example: $\widetilde{X} = [0, 2\pi] \subset \Re = X$ and $y = f(x) = \sin(x)$. Hence, $f\left(\widetilde{X}\right) = [-1, 1] \subset \Re = Y$.

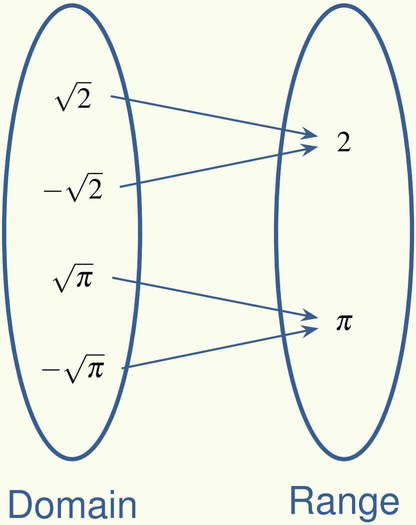
Domain and Range (Image)

- Let $T : U \longrightarrow V$ be a map.
- U is called the **domain** of T and V the **codomain**.
- The **range** or **image** of T, denoted by $\mathrm{range}(T)$, is the set of all possible outputs in the codomain.
- Mathematically, $\mathrm{range}(T) = \mathrm{image}(T) = \{T(x) : x \in U\}$.
- For example, if T is given by $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ for some matrix $\boldsymbol{A}$, then the range of T is given by the **span (set of all possible linear combinations) of A's column vectors**.

Surjection

- T is said to be **surjective** (or **onto**) if its range equals the codomain.
- It means that every vector in the codomain can be the output of T.
- If T is surjective, it is called a **surjection**.
- Example
  - Let $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$.
  - Note that every vector in $\mathrm{range}(T)$ is of the form $\begin{bmatrix} a \\ -a \end{bmatrix}$.
  - Hence, T is not surjective since there are vectors in $\mathfrak{R}^2$ that are not in the range of T, e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Illustration of Surjection



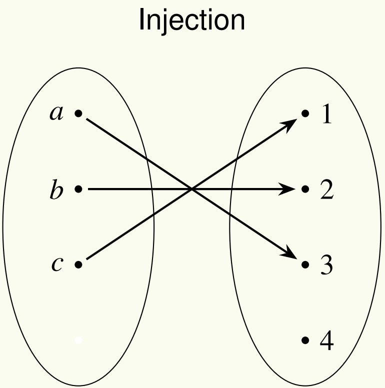
Injection and Bijection

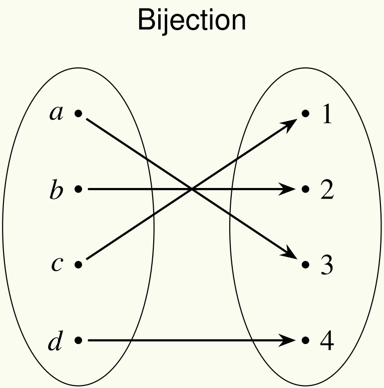
- T is said to be **injective** (or **one-to-one**) if for all distinct $\boldsymbol{x}, \boldsymbol{y} \in U$, $T(\boldsymbol{x}) \neq T(\boldsymbol{y})$.
- It means that different inputs lead to different outputs.
- If T is injective, it is called an **injection**.
- Example
  - Consider the same T in the example above. It is not injective because for every $a \in \mathfrak{R}$,

$$T\left(\begin{bmatrix} a \\ a \end{bmatrix}\right) = \begin{bmatrix} a-a \\ -a+a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- If T is both surjective and injective, it is said to be **bijective** and we call T a **bijection**.

Illustration of Injection and Bijection





Some Important Maps

- The **identity map**: $f(x) = x$ for all $x \in X$.
- A **composite map**: Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. Then $g \circ f : X \longrightarrow Z$ is a composite map. In other words,
$$(g \circ f)(x) = g\big(f(x)\big).$$
- The **inverse map**: Let $f : X \longrightarrow Y$. Then for $y \in Y$, the set of $x \in X$ that satisfies $f(x) = y$ is called the inverse image, and it is represented by $f^{-1}(y)$. That is
$$f^{-1}(y) = \Big\{ x \in X \Big| f(x) = y \Big\}.$$
- If $f : X \longrightarrow Y$ is bijective, there is only one inverse image that corresponds to every $y \in Y$. Then we have the inverse map from Y back unto X, and it is expressed as f^{-1}, so that $x = f^{-1}(y)$. Moreover, $f^{-1} : Y \longrightarrow X$ is bijective too.

Illustration of Composite Map

- A **composite map**: $(g \circ f)(x) = g\big(f(x)\big)$.

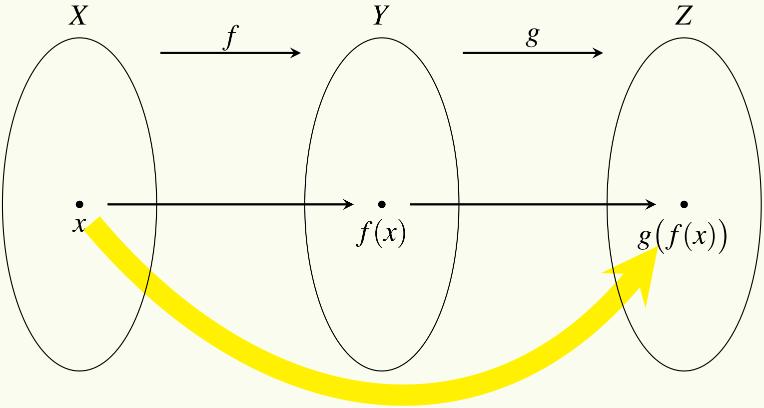
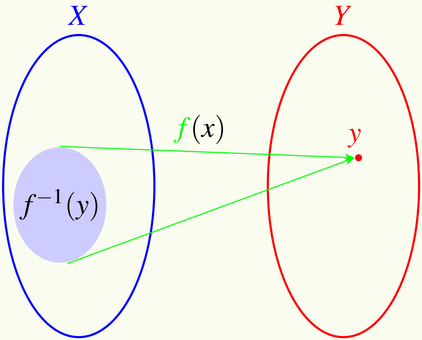


Illustration of Inverse Map

- The **inverse map** $f^{-1}(y) = \left\{ x \in X \middle| f(x) = y \right\}$.



Definition of Linear Map

- Let V and W be vector spaces.
- A map $T : V \longrightarrow W$ is a **linear transformation** if for any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in V and any scalar $c \in \Re$, the following are satisfied:

$$T(\boldsymbol{x}+\boldsymbol{y}) = T(\boldsymbol{x})+T(\boldsymbol{y}) \tag{1}$$

$$T(c\boldsymbol{x}) = cT(\boldsymbol{x}) \tag{2}$$

Matrix as Linear Map

Theorem 3.1.

Let $\boldsymbol{A}$ be a $m \times n$ matrix. Consider a map g from $\Re^n$ to $\Re^m$ determined by the matrix $\boldsymbol{A}$:

$$g : \boldsymbol{x} \longrightarrow \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \quad \left(\boldsymbol{x} \in \Re^n; \boldsymbol{y} \in \Re^m\right).$$

The map g is a linear map.

Proof.

- For $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Re^n$, we have $\boldsymbol{y}_1 = g(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = g(\boldsymbol{x}_2)$.
- Now, $g(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2 = \boldsymbol{y}_1 + \boldsymbol{y}_2$.
- Hence, (1) is satisfied.
- Next, let c be the scalar. (2) is satisfied as well:

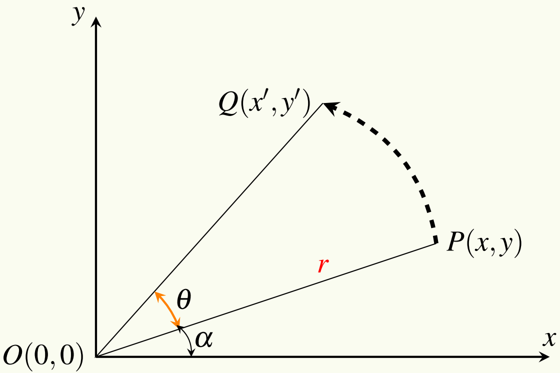
$$g(c\boldsymbol{x}) = \boldsymbol{A}(c\boldsymbol{x}) = c(\boldsymbol{A}\boldsymbol{x}) = cg(\boldsymbol{x}).$$

□

Example: Rotation Matrix

Example 3.2.

A point $P(x,y)$ on a plane is rotated θ degrees anticlockwise with respect to the origin $O(0,0)$. Is $\begin{bmatrix} x' \\ y' \end{bmatrix} = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ a linear map?



Answer to Example 3.2

- In the polar coordinate system, x and y can be expressed in terms of the radius r and the angle α:

$$x = r\cos(\alpha), \qquad y = r\sin(\alpha).$$

- It follows that

$$\begin{aligned}\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} = \begin{bmatrix} r\cos(\alpha)\cos(\theta) - r\sin(\alpha)\sin(\theta) \\ r\sin(\theta)\cos(\alpha) + r\cos(\theta)\sin(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta)x - \sin(\theta)y \\ \sin(\theta)x + \cos(\theta)y \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.\end{aligned}$$

- According to Theorem 3.1, $f : \Re^2 \longrightarrow \Re^2$ is a linear map through the **rotation matrix** $\boldsymbol{R}(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Example: Projected Rotation

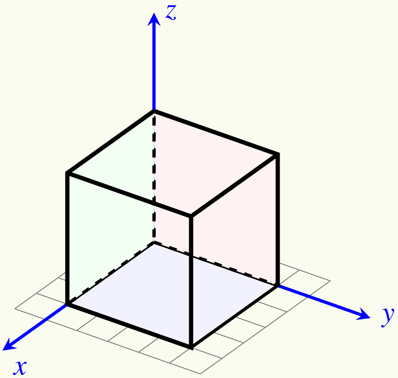
Example 3.3.

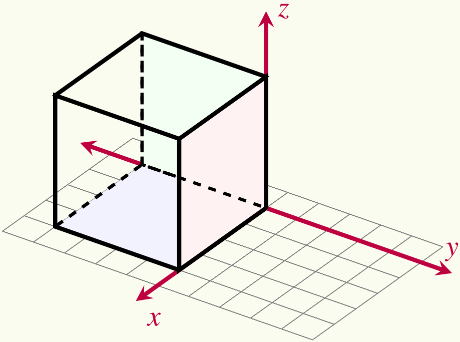
- Let (a, b, c) be the coordinates of a point P in $\mathfrak{R}^3$.
- Then, we define $\boldsymbol{u}_1 = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$, and $\boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Let us define a point Q in $\mathfrak{R}^2$ by

$$\overrightarrow{OQ} = a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3.$$

- Let (p, q) be the coordinates of point Q.
- Is the map from (a, b, c) to (p, q) linear?

Illustration of Projected Rotation





Answer to Example 3.3

- In the matrix form, the relationship between (a,b,c) and (p,q) is given by

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \cos(\beta) & 0 \\ \sin(\alpha) & \sin(\beta) & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- Define $\boldsymbol{A} := \begin{bmatrix} \cos(\alpha) & \cos(\beta) & 0 \\ \sin(\alpha) & \sin(\beta) & 1 \end{bmatrix}$.
- It follows that

$$\begin{bmatrix} p \\ q \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

- Applying Theorem 3.1, we conclude that the map $f : \Re^3 \longrightarrow \Re^2$ is a linear map.

Isomorphic Map

Definition 3.4.

Let f be a linear map from the linear space U to another linear space V. Moreover f is assumed to be bijective. When these conditions are fulfilled, f is said to be an **isomorphic map**, and U and V are said to be **isomorphic**.

Theorem 3.5.

When the linear spaces U and V have the same dimension, then there exists an **isomorphic map** *that connects these two spaces.*

Proof of Theorem 3.5

- Let $\dim U = \dim V = n$, and let their respective basis be $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$.
- U being a linear space, we can write $\boldsymbol{x} = c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n$.
- Let $f : U \longrightarrow V$ be a linear map, so that $f(\boldsymbol{u}_i) = \boldsymbol{v}_i$, where $i = 1, \ldots, n$. Consequently, $f(\boldsymbol{x}) = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$.
- Suppose $U \ni \boldsymbol{y} = d_1\boldsymbol{u}_1 + \cdots + d_n\boldsymbol{u}_n$. Then

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{y}) &= (c_1+d_1)\boldsymbol{v}_1 + \cdots + (c_n+d_n)\boldsymbol{v}_n \\ &= \big(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n\big) + \big(d_1\boldsymbol{v}_1 + \cdots + d_n\boldsymbol{v}_n\big) \\ &= f(\boldsymbol{x}) + f(\boldsymbol{y}). \end{aligned}$$

- Also, for $a \in \Re$,
$f(a\boldsymbol{x}) = ac_1\boldsymbol{v}_1 + \cdots + ac_n\boldsymbol{v}_n = a\big(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n\big) = af(\boldsymbol{x})$.

Proof of Theorem 3.5 (Cont'd)

- For all $\boldsymbol{x} \in U$, the range $\{f(\boldsymbol{x}) | \boldsymbol{x} \in U\}$ is V. So f is surjective.
- Moreover, we let $f(\boldsymbol{x}) = f(\boldsymbol{y})$. By definition, $c_1 = d_1, \ldots, c_n = d_n$, and it follows that $\boldsymbol{x} = \boldsymbol{y}$, which means that f is injective.
- Being both surjective and injective, f is therefore bijective.
- The conditions in Definition 3.4 are satisfied, which means that f is isomorphic.

$\square$

Example of an Isomorphic Map

Example 3.6.

- Consider the set of quadratic functions:
$$\mathbb{R}[x]_2 = \{a_0 + a_1 x + a_2 x^2 \,|\, a_0, a_1, a_2 \in \Re\}.$$
- The standard basis of $\mathbb{R}[x]_2$ is $\{1, x, x^2\}$.
- It is clear that $\dim \mathbb{R}[x]_2 = \dim \Re^3 = 3$.
- By Theorem 3.5, there exists an isomorphic map that connects $\mathbb{R}[x]_2$ and $\Re^3$.
- Concretely, $p = s + tx + ux^2 \in \mathbb{R}[x]_2$ and
$$f(p) = s\boldsymbol{e}_1 + t\boldsymbol{e}_2 + u\boldsymbol{e}_3 = \begin{bmatrix} s \\ t \\ u \end{bmatrix} \in \Re^3.$$

Representation Matrix of a Map

Definition 3.7 (Representation Matrix of a Map).

- Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ be the respective basis of linear spaces U and V, and $f : U \longrightarrow V$ is a linear map.
- When $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are mapped by f, they become the elements of V and thus can be written as the linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$. That is,

$$\begin{aligned} f(\boldsymbol{u}_1) &= a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + \cdots + a_{m1}\boldsymbol{v}_m, \\ f(\boldsymbol{u}_2) &= a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + \cdots + a_{m2}\boldsymbol{v}_m, \\ &\vdots \\ f(\boldsymbol{u}_n) &= a_{1n}\boldsymbol{v}_1 + a_{2n}\boldsymbol{v}_2 + \cdots + a_{mn}\boldsymbol{v}_m. \end{aligned}$$

- Let $\boldsymbol{A} = \left[a_{ij}\right]_{m\times n}$. It is called the **representation matrix** with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$, and the **matrix corresponding to the linear map f**.

Example

Example 3.8.

Find the matrix $\boldsymbol{A}$ that corresponds to the map f in Example 3.2.

- The standard basis of $\Re^2$ is $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Accordingly,

$$f(\boldsymbol{e}_1) = \cos(\theta)\boldsymbol{e}_1 + \sin(\theta)\boldsymbol{e}_2, \quad f(\boldsymbol{e}_2) = -\sin(\theta)\boldsymbol{e}_1 + \cos(\theta)\boldsymbol{e}_2.$$

- It follows that

$$\begin{bmatrix} f(\boldsymbol{e}_1) & f(\boldsymbol{e}_2) \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

- Thus, we have found $\boldsymbol{A}$ to be the rotation matrix $\boldsymbol{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ in Example 3.2.

What is image? What is kernel?

Let f be a linear map from the linear space U to another linear space V.

- The **image** of f is the set

$$\operatorname{Im} f := \{ f(\boldsymbol{u}) | \boldsymbol{u} \in U \} \quad (= f(U)),$$

  which is a subspace of V.

- The **kernel** is the set

$$\operatorname{Ker} f := \{ \boldsymbol{u} \in U | f(\boldsymbol{u}) = \mathbf{0} \},$$

  which is a subspace of U.

Examples of Image and Kernel

Image

- In Example 3.2, all the points of $\Re^2$ can be rotated. Therefore $\mathrm{Im} f = \{(x,y)|x,y \in \Re\} = \Re^2$.
- All the points of $\Re^2$ can be mapped to the x axis by a linear map f. Therefore, $\mathrm{Im}\, f = \left\{(x,0)\middle|x \in \Re\right\}$.

Kernel

- In Example 3.2, only the origin can be mapped to the origin. Therefore $\mathrm{Ker} f = \{(0,0)\}$.
- All the points of the y axis are mapped to the origin but not the other points in $\Re^2$. Therefore, $\mathrm{Ker} f = \left\{(0,y)|f(0,y) = (0,0), \forall y \in \Re\right\}$.

Theorems

- $\mathrm{Ker}f = \{\mathbf{0}\} \Longleftrightarrow f$ is bijective.
- $\dim \mathrm{Ker}f + \dim \mathrm{Im}f = \dim U$
- With respect to $m \times n$ matrix $\boldsymbol{A}$ and $n \times \ell$ matrix $\boldsymbol{B}$,

$$\mathrm{rank}\,\boldsymbol{AB} \leqq \mathrm{rank}\,\boldsymbol{A}, \quad \text{and} \quad \mathrm{rank}\,\boldsymbol{AB} \leqq \mathrm{rank}\,\boldsymbol{B}.$$

- With respect to $m \times n$ matrix $\boldsymbol{A}$, m-dimensional regular square matrix $\boldsymbol{P}$, and n-dimensional regular square matrix $\boldsymbol{Q}$,

$$\mathrm{rank}\,\boldsymbol{PA} = \mathrm{rank}\,\boldsymbol{A} = \mathrm{rank}\,\boldsymbol{AQ}.$$

- Let $\boldsymbol{A}$ be the matrix that corresponds to the linear map $f : \mathfrak{R}^n \longrightarrow \mathfrak{R}^m$. Then
  1. $\mathrm{rank}\,\boldsymbol{A} = n \Longleftrightarrow f$ is injective.
  2. $\mathrm{rank}\,\boldsymbol{A} = m \Longleftrightarrow f$ is surjective.

Theorem on Basis Transformation

Theorem 5.1.

- *Let* $\boldsymbol{P}$ *be the basis transformation matrix for the* n*-dimensional linear space* U*. It transforms the basis* $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ *to another basis* $\{\widetilde{\boldsymbol{u}}_1, \widetilde{\boldsymbol{u}}_2, \ldots, \widetilde{\boldsymbol{u}}_n\}$*.*
- *Likewise,* $\boldsymbol{Q}$ *denotes the basis transformation matrix for the* m*-dimensional linear space* V*. It transforms the basis* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ *to another basis* $\{\widetilde{\boldsymbol{v}}_1, \widetilde{\boldsymbol{v}}_2, \ldots, \widetilde{\boldsymbol{v}}_m\}$*.*
- f *is a linear map* $f : U \longrightarrow V$ *with* $\boldsymbol{A}$ *being the corresponding representation matrix that transforms the basis of* U*,* $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$*, to the basis of* V*,* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$*.*
- *Then, the matrix* $\widetilde{\boldsymbol{A}}$ *that corresponds to the transformation of* $\{\widetilde{\boldsymbol{u}}_1, \widetilde{\boldsymbol{u}}_2, \ldots, \widetilde{\boldsymbol{u}}_n\}$ *to* $\{\widetilde{\boldsymbol{v}}_1, \widetilde{\boldsymbol{v}}_2, \ldots, \widetilde{\boldsymbol{v}}_m\}$ *is*

$$\widetilde{\boldsymbol{A}} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{P}.$$

Isomorphic Map and Regularity of Matrix

Theorem 5.2.

Let $\boldsymbol{P}$ be the basis transformation matrix for the n-dimensional linear space U. It transforms the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ to another basis $\{\widetilde{\boldsymbol{u}}_1, \widetilde{\boldsymbol{u}}_2, \ldots, \widetilde{\boldsymbol{u}}_n\}$. Let $f : U \longrightarrow U$ be a map with $\boldsymbol{A}$ being the corresponding matrix with respect to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$. The matrix that corresponds to the map f with respect to the basis $\{\widetilde{\boldsymbol{u}}_1, \widetilde{\boldsymbol{u}}_2, \ldots, \widetilde{\boldsymbol{u}}_n\}$ is $\widetilde{\boldsymbol{A}}$. Then

$$\widetilde{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}.$$

Theorem 5.3.

Linear map f being isomorphic is equivalent to the matrix $\boldsymbol{A}$ that corresponds to f being regular.

Takeaways

- A map is a generalized version of function.
- A linear map from a space to another space provides a geometric meanings such as rotation and projection to its corresponding matrix that is closed under addition and scalar multiplication, i.e., linear combination.
- The inverse map is unique only for the bijective map.
- Isomorphism refers to two linear spaces having a bijection connecting every point of one space to the corresponding point in the other space.
- The kernel is in the input space, and the image is in the output space.
- Isomorphic map is equivalent to a regular matrix that corresponds to it.

Keywords