Lesson 5
Determinant

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Learning Outcomes

- Explain the reasons for why we need to study determinant.
- Elaborate important vocabulary and concepts such as determinant, cyclic, interchange, identity, inverse, even, and odd permutations, sign of permutation, co-factor, minor, adjoint matrix, Cramer's formula.
- Connect the key concepts and results to the takeaways of the previous lessons, especially with regard to the system of linear equations, Gauss-Jordan elimination method, basic row operations, and so on.
- Determine whether a square matrix is regular or not.
- Solve the problems that involve the system of linear equations, which requires the computation of determinant.

What is a determinant?

Definition 1.1 (Two-Dimensional Matrix's Determinant).

With respect to $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we define the **2-dimensional determinant** as

$$\det \boldsymbol{A} = a_{11}a_{22} - a_{12}a_{21},$$

which is a scalar, and we write

$$\det \boldsymbol{A} =: \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Note that the inverse matrix is

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Geometrical Interpretation

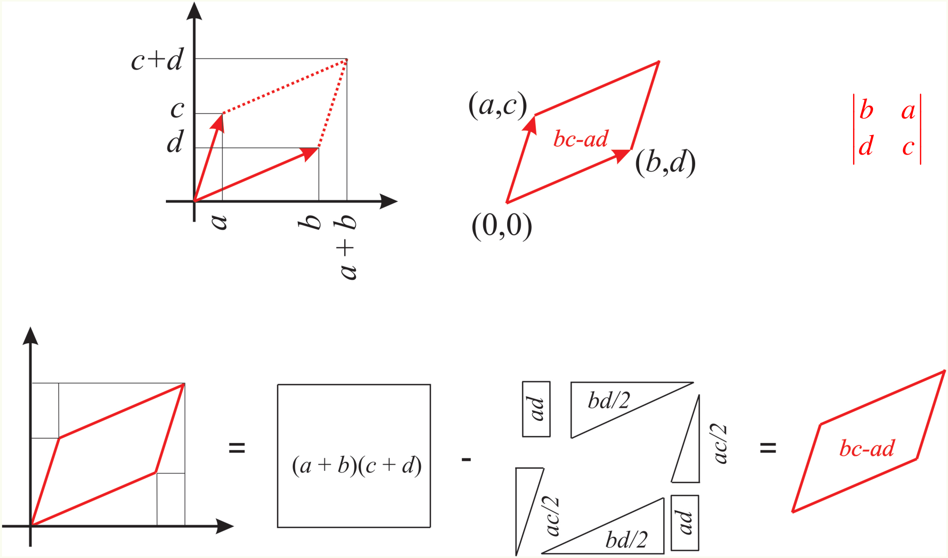
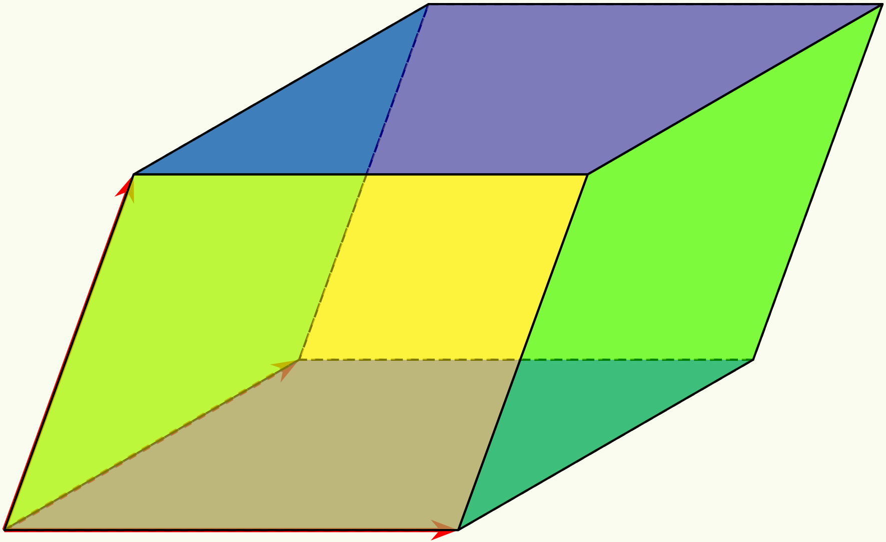


Illustration of 3-Dimensional Determinant



Simple Solution

- Two-variable system of linear equations

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

- The solution is $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$, which can be written in terms of only the determinants.

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Humor

On this quiz, I asked you to find the determinant of a 2x3 matrix. Some of you, to my great amusement, actually tried to do this.

— **Math Professor Quote**

Definition of Permutation

Definition 2.1 (Permutation).

Let $S := \{1, 2, \ldots, n\}$ be a set of n natural numbers. A **permutation** is defined as a 1-to-1 mapping σ from S to S such that

$$\sigma(1) = k_1,\ \sigma(2) = k_2,\ \ldots,\ \sigma(n) = k_n,$$

where $\{k_1, k_2, \ldots, k_n\}$ is a re-arranged version of set S. Permutation at times is expressed as

$$\sigma := \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}. \tag{1}$$

Definition 2.2 (Product of Two Permutations).

Let σ and τ be two permutations on S. The product $\sigma\tau(i)$ is defined as $\sigma\big(\tau(i)\big)$ for $i = 1, 2, \ldots, n$.

Examples

(1) When $\sigma(1)=3, \sigma(2)=2, \sigma(3)=4$, and $\sigma(4)=1$, we write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1 \\ 1 & 4 & 3 \end{pmatrix}.$$

(2) Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

- $\sigma\tau$ and $\tau\sigma$ are given by

$$\begin{aligned} \sigma(\tau(1)) &= \sigma(3) = 1 & \tau(\sigma(1)) &= \tau(2) = 2, \\ \sigma(\tau(2)) &= \sigma(2) = 3 & \tau(\sigma(2)) &= \tau(3) = 1, \\ \sigma(\tau(3)) &= \sigma(1) = 2 & \tau(\sigma(3)) &= \tau(1) = 3. \end{aligned}$$

- Thus, $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3. \end{pmatrix}$
- Takeaway: $\sigma\tau \neq \tau\sigma$, i.e. non-commutative.

Identity, Inverse, and Cyclic Permutations

Definition 2.3 (Identity and Inverse Permutation).

The **identity permutation** is defined as $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$. The **inverse permutation** is defined as $\sigma^{-1} := \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ 1 & 2 & \cdots & n \end{pmatrix}$.

Definition 2.4 (Cyclic Permutation).

From a set of n numbers $\{1, 2, \ldots, n\}$, take $r \leq n$ different numbers denoted by $k_1, k_2, \ldots, k_r$. The **cyclic permutation** is defined as

$$\sigma := \begin{pmatrix} k_1 & k_2 & \cdots & k_{r-1} & k_r \\ k_2 & k_3 & \cdots & k_r & k_1 \end{pmatrix}.$$

That is, the sequence of r different numbers is shifted by one position to the right.

Examples

(1) The inverse permutation of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ is

$$\sigma^{-1} = \begin{pmatrix} 4 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

(2) Suppose $\sigma(2) = 5, \sigma(5) = 3, \sigma(3) = 2$, and the other numbers do not permute. Then we can write the cyclic permutation simply as $\begin{pmatrix} 5 & 2 & 3 \end{pmatrix}$. It can also be written as $\begin{pmatrix} 2 & 3 & 5 \end{pmatrix}$, or as $\begin{pmatrix} 3 & 5 & 2 \end{pmatrix}$.

(3) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 & 8 \end{pmatrix}$.

- Numbers 1 and 8 do not permute.
- Since $\sigma(2) = 3, \sigma(3) = 5, \sigma(5) = 2$, we have a cyclic permutation $(2\ \ 3\ \ 5)$.
- Since $\sigma(4) = 7, \sigma(7) = 6, \sigma(6) = 4$, we have another cyclic permutation $(4\ \ 7\ \ 6)$.
- Hence $\sigma = (2\ \ 3\ \ 5)(4\ \ 7\ \ 6)$.

Interchanges and Sign

Definition 2.5 (Interchange Permutation).

An **interchange** is defined as the cyclic permutation for 2 elements in the set S. while keeping other elements unchanged. We denote the interchange of i and j as $(i\,j)$.

Theorem 2.6 (Product of Cyclic Permutations or Interchanges).

1. *Any permutation can be expressed as the product of cyclic permutations.*
2. *Any permutation can be expressed as the product of interchanges.*

Definition 2.7 (Sign of Permutation).

When the permutation σ is expressed as a product of m interchanges, the **sign** of the permutation is defined as

$$\mathrm{sgn}(\sigma) := (-1)^m.$$

The sign of the identity permutation is 1 (i.e., $m = 0$).

Even and Odd Permutations

Definition 2.8 (Even and Odd Permutations).

When $\mathrm{sgn}(\sigma) = 1$, the permutation is said to be **even**. On the other hand, if $\mathrm{sgn}(\sigma) = -1$, the permutation is said to be **odd**.

Theorem 2.9 (Uniqueness of Sign of a Permutation).

When the permutation σ is expressed as $\sigma := \tau_1 \cdots \tau_k$, its sign does not depend on whether k is even or odd. The sign of σ is uniquely determined.

Definition 2.10 (Set of All Permutations).

The **set of all possible permutations** involving n numbers is denoted by S_n.

Cyclic Permutation as a Product of Interchanges

- Note that for any interchange, $(i \;\; j) = (j \;\; i)$

- Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

★ As a product of 3 interchanges: $\sigma = (1 \;\; 2)(1 \;\; 3)(1 \;\; 4)$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 1 & 1 \\ & \downarrow & \downarrow & \\ & 3 & 4 & \end{array}$$

★ $\text{sgn}(\sigma) = (-1)^3 = -1$.

- Takeaways: the sign of cyclic permutation with even number of elements is -1, and that with odd number is 1.

Definition of Determinant

Definition 3.1 (Definition of a Generic Determinant).

Given a n-dimensional square matrix $\boldsymbol{A} = \left[a_{ij}\right]$, its **determinant** is defined as

$$\det \boldsymbol{A} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

- Other notations of determinant:

$$|\boldsymbol{A}|, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}, \quad \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det \begin{bmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_n' \end{bmatrix}, \quad \begin{vmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_n' \end{vmatrix},$$

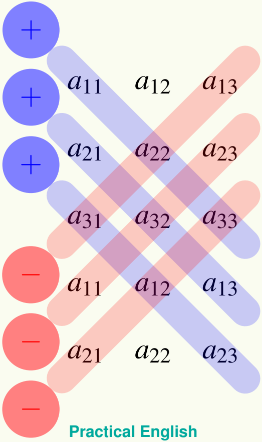
$$\begin{vmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{vmatrix},$$

where $\boldsymbol{a}_i, i = 1, 2, \ldots, n$, is a n-dimensional vector.

Illustration of Determinant

$$\det\left(\boldsymbol{A}\right)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$=\left(a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}\right)-\left(a_{13}a_{22}a_{31}+a_{23}a_{32}a_{11}+a_{33}a_{12}a_{21}\right)$$



Another Illustration

$$\boldsymbol{A}_{3\times 3} = \begin{array}{c} \begin{matrix} + & \quad + & \quad +\,- & \quad - & \quad - \end{matrix} \\ \left[\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right] \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \end{array}$$

$$\det \boldsymbol{A} = ({\color{blue}a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3}) - ({\color{red}a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1})$$

Theorem 3.2

Theorem 3.2.

The following equations are true.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & 0 \\ \vdots & \ddots & \vdots & 0 \\ a_{n-1\,1} & \cdots & a_{n-1\,n-1} & \vdots \\ a_{n1} & \cdots & a_{n\,n-1} & a_{nn} \end{vmatrix} = a_{nn} \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1\,1} & \cdots & a_{n-1\,n-1} \end{vmatrix}$$

Theorems 3.3 and 3.4

Theorem 3.3 (Invariance under Transpose).

For a square matrix $\boldsymbol{A}$,

$$\det \boldsymbol{A}' = \det \boldsymbol{A}.$$

Theorem 3.4 (Scalar Multiple and Addition).

Let $\boldsymbol{a}_i$ and $\boldsymbol{b}_i, i = 1, 2, \ldots n$ be 1-dimensional vectors and c a scalar. Then

$$\begin{vmatrix} \boldsymbol{a}'_1 \\ \vdots \\ c\boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}'_n \end{vmatrix} = c \begin{vmatrix} \boldsymbol{a}'_1 \\ \vdots \\ \boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}'_n \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \boldsymbol{a}'_1 \\ \vdots \\ \boldsymbol{a}'_i + \boldsymbol{b}'_i \\ \vdots \\ \boldsymbol{a}'_n \end{vmatrix} = \begin{vmatrix} \boldsymbol{a}'_1 \\ \vdots \\ \boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}'_n \end{vmatrix} + \begin{vmatrix} \boldsymbol{a}'_1 \\ \vdots \\ \boldsymbol{b}'_i \\ \vdots \\ \boldsymbol{a}'_n \end{vmatrix}.$$

Theorem 3.5

Theorem 3.5 (Interchangeability).

$$(1)\quad \begin{vmatrix} \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_j' \\ \vdots \end{vmatrix} = -\begin{vmatrix} \vdots \\ \boldsymbol{a}_j' \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \end{vmatrix}, \qquad (2)\quad \begin{vmatrix} \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \end{vmatrix} = 0.$$

- Swapping one row with the other results in only a sign change.

Theorem 3.6

Theorem 3.6 (Invariance under a Basic Row Operation).

$$\begin{vmatrix} \vdots \\ \boldsymbol{a}_i' + c\boldsymbol{a}_j' \\ \vdots \\ \boldsymbol{a}_j' \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_j' \\ \vdots \end{vmatrix}$$

Example: Application of Theorems

$$\begin{aligned}
\begin{vmatrix} -4 & 1 & 1 \\ 1 & 2 & 0 \\ -1 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} -4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} && \left(R_3 \leftarrow R_3 + R_2 \text{ [Th 3.6]}\right) \\
&= \begin{vmatrix} 0 & 9 & 1 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} && \left(R_1 \leftarrow R_1 + 4R_2 \text{ [Th 3.6]}\right) \\
&= \begin{vmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} && \left(R_1 \leftarrow R_1 - 3R_3 \text{ [Th 3.6]}\right) \\
&= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 3 & 1 \end{vmatrix} && \left(R_1 \leftrightarrow R_2 \text{ [Th 3.5]}\right) \\
&= -\left(-\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix}\right) && \left(R_2 \leftrightarrow R_3 \text{ [Th 3.5]}\right) \\
&= \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{vmatrix} = 1 \times 3 \times (-2) = -6. && \left(\text{[Th 3.2]}\right)
\end{aligned}$$

Theorem 3.7 to Theorem 3.9

Theorem 3.7 (Determinants of Basic Operation Matrices).

For the basic operation matrices, scalars c and d,

$$\left|\boldsymbol{P}_{ij}\right| = -\left|\boldsymbol{I}_n\right| = -1, \qquad \left|\boldsymbol{P}_{ij}\boldsymbol{A}\right| = \left|\boldsymbol{P}_{ij}\right|\left|\boldsymbol{A}\right|,$$

$$\left|\boldsymbol{Q}_i(c)\right| = c\left|\boldsymbol{I}_n\right| = c, \qquad \left|\boldsymbol{Q}_i(c)\boldsymbol{A}\right| = \left|\boldsymbol{Q}_i(c)\right|\left|\boldsymbol{A}\right|,$$

$$\left|\boldsymbol{R}_{ij}(d)\right| = \left|\boldsymbol{I}_n\right| = 1, \qquad \left|\boldsymbol{R}_{ij}(d)\boldsymbol{A}\right| = \left|\boldsymbol{R}_{ij}(d)\right|\left|\boldsymbol{A}\right|.$$

Theorem 3.8 (Regularity and Non-Zero Determinant).

An n-dimensional square matrix $\boldsymbol{A}$ is regular if and only if $\det \boldsymbol{A} \neq 0$.

Theorem 3.9 (Interchangeability with Multiplication).

For the n-dimensional matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\left|\boldsymbol{A}\boldsymbol{B}\right| = \left|\boldsymbol{A}\right|\left|\boldsymbol{B}\right|.$$

Determinants of Partitioned Matrices

Theorem 3.10 (With Null and Identity Sub-Matrices).

Let $\boldsymbol{A}$ be an n-dimensional matrix.

(1) *If* $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_{12} \\ \mathbf{0} & \boldsymbol{A}_{22} \end{bmatrix}$, *then* $\det \boldsymbol{A} = \det \boldsymbol{A}_{22}$.

(2) *If* $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \mathbf{0} \\ \boldsymbol{A}_{21} & \boldsymbol{I} \end{bmatrix}$, *then* $\det \boldsymbol{A} = \det \boldsymbol{A}_{11}$.

Theorem 3.11 (With Null Sub-Matrice Only).

If an n-dimensional square matrix can be partitioned as $\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \mathbf{0} & \boldsymbol{A}_{22}. \end{bmatrix}$, *or* $\begin{bmatrix} \boldsymbol{A}_{11} & \mathbf{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}. \end{bmatrix}$, *then*

$$\det \boldsymbol{A} = \left(\det \boldsymbol{A}_{11}\right)\left(\det \boldsymbol{A}_{22}\right).$$

Minor of a Matrix

- The **minors** of $\boldsymbol{A}$, each of size $(n-1)\times(n-1)$, is a resulting matrix $\widetilde{\boldsymbol{A}}_{ij}$, after column j and row i have been deleted.

$$
\begin{array}{cc}
 & \begin{array}{c}\text{column}\\ j\\ \downarrow\end{array} \\
\widetilde{\boldsymbol{A}}_{ij} := \begin{bmatrix}
a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1\,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\
 & & & & & \\
a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{n\,1} & \cdots & a_{n\,j-1} & a_{n\,j+1} & \cdots & a_{n\,n}
\end{bmatrix} & \leftarrow \text{Row } i
\end{array}
$$

Co-Factor and Expansion

Definition 4.1 (Co-Factor).

Given an n-dimensional square matrix $\boldsymbol{A} = [a_{ij}]$, its (ij) **co-factor** $\widetilde{a}_{ij}$ is a scalar defined by the following formula:

$$\widetilde{a}_{ij} := (-1)^{i+j} \det \widetilde{\boldsymbol{A}}_{ij}.$$

Theorem 4.2 (Expansion with Co-Factors).

For an n-dimensional matrix $\boldsymbol{A}$, its determinant can be written as an expansion of its co-factors with respect to the i row

$$\det \boldsymbol{A} = a_{i1}\widetilde{a}_{i1} + \cdots + a_{in}\widetilde{a}_{in}, \qquad \text{where } i = 1, 2, \ldots, n,$$

or with respect to the j column

$$\det \boldsymbol{A} = a_{1j}\widetilde{a}_{1j} + \cdots + a_{1j}\widetilde{a}_{nj}, \qquad \text{where } j = 1, 2, \ldots, n. \tag{2}$$

Example 4.3

Example 4.3 (Expansion).

Compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & -4 \end{bmatrix}$ by expanding along the second column.

Answer:

$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & -4 \end{vmatrix} = (-1)^{1+2}(0)\begin{vmatrix} 3 & 5 \\ 2 & -4 \end{vmatrix} + (-1)^{2+2}(4)\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}$$

$$+(-1)^{3+2}(-1)\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

$$= -32 + (-1) = -33.$$

Co-Factor Matrix

Definition 4.4 (Co-Factor or Adjoint Matrix).

Given an n-dimensional square matrix $\boldsymbol{A} = [a_{ij}]$, we define $\widetilde{\boldsymbol{A}} := [\widetilde{a}_{ij}]$ and call its transpose $\widetilde{\boldsymbol{A}}'$ as the **co-factor matrix** or the **adjoint matrix**.

Theorem 4.5 (Product with Co-Factor Matrix).

For an n-dimensional square matrix $\boldsymbol{A}$ and its co-factor $\widetilde{\boldsymbol{A}}'$, the following equations hold

$$\boldsymbol{A}\widetilde{\boldsymbol{A}}' = \widetilde{\boldsymbol{A}}'\boldsymbol{A} = (\det \boldsymbol{A})\boldsymbol{I}.$$

Theorem 4.6 (Inverse of a Matrix).

If an n-dimensional square matrix $\boldsymbol{A}$ is regular, then its inverse is given by

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\widetilde{\boldsymbol{A}}'.$$

Example 4.7

Example 4.7 (Adjoint Matrix and Inverse).

Find the adjoint matrix of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $\boldsymbol{A}^{-1}$.

- The co-factors are

$$\widetilde{a}_{11} = (-1)^{1+1} \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -7, \widetilde{a}_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1, \widetilde{a}_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 4,$$

$$\widetilde{a}_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1, \widetilde{a}_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1, \widetilde{a}_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$

$$\widetilde{a}_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} = 11, \widetilde{a}_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2, \widetilde{a}_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = -5.$$

Example 4.7 (Cont'd)

- Consequently, the adjoint matrix $\boldsymbol{A}^\dagger \equiv \widetilde{\boldsymbol{A}}'$ is

$$\boldsymbol{A}^\dagger = \begin{bmatrix} -7 & -1 & 4 \\ -1 & -1 & 1 \\ 11 & 2 & -5 \end{bmatrix}' = \begin{bmatrix} -7 & -1 & 11 \\ -1 & -1 & 2 \\ 4 & 1 & -5 \end{bmatrix}.$$

- The determinant of $\boldsymbol{A}$ is

$$\begin{aligned} \det(\boldsymbol{A}) &= 1\cdot(-3)\cdot 2 + 1\cdot 1\cdot 3 + 2\cdot 1\cdot 1 - \big(3\cdot(-3)\cdot 1 + 1\cdot 1\cdot 1 + 2\cdot 1\cdot 2\big) \\ &= -1-(-4) = 3. \end{aligned}$$

- Therefore, from Theorem 4.6,

$$\boldsymbol{A}^{-1} = \frac{1}{\det\boldsymbol{A}}\boldsymbol{A}^\dagger = \frac{1}{3}\begin{bmatrix} -7 & -1 & 11 \\ -1 & -1 & 2 \\ 4 & 1 & -5 \end{bmatrix}.$$

Cramer's Formula

Theorem 4.8 (Cramer's Formula).

To solve the simultaneous first-order system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *for which the coefficient matrix* $\boldsymbol{A}$ *is regular, we have*

$$x_j = \frac{\det \boldsymbol{B}_j}{\det \boldsymbol{A}}, \tag{3}$$

where the matrix $\boldsymbol{B}_j$ *is given by*

$$\boldsymbol{B}_j := \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{j-1} & \boldsymbol{b} & \boldsymbol{a}_{j+1} & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$

That is, column j *of* $\boldsymbol{A}$ *is replaced by* $\boldsymbol{b}$*.*

Proof of Cramer's Formula

- Since $\boldsymbol{A}$ is regular, $\boldsymbol{A}^{-1}$ exists. Multiply $\boldsymbol{A}^{-1}$ from the left to both sides of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ to obtain the solution: $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.
- Applying Theorem 4.6, we obtain $\boldsymbol{x} = \dfrac{1}{\det \boldsymbol{A}} \widetilde{\boldsymbol{A}}' \boldsymbol{b}$.
- The j-th element of the solution $\boldsymbol{x}$ is calculated as

$$x_j = \frac{1}{\det \boldsymbol{A}} \begin{bmatrix} \widetilde{a}_{1j} & \widetilde{a}_{2j} & \cdots & \widetilde{a}_{nj} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \frac{1}{\det \boldsymbol{A}} \left(b_1 \widetilde{a}_{1j} + b_2 \widetilde{a}_{2j} + \cdots + b_n \widetilde{a}_{nj} \right).$$

- With reference to (2), we obtain a co-factor expansion $b_1 \widetilde{a}_{1j} + b_2 \widetilde{a}_{2j} + \cdots + b_n \widetilde{a}_{nj}$, of the determinant of $\boldsymbol{B}_j$. Hence, we arrive at (3) and the proof is complete. $\square$

Computing Determinants by Row Reduction

- Let $\boldsymbol{A}$ be a square matrix. After some number of row operations on $\boldsymbol{A}$, the row echelon matrix $\boldsymbol{B}$ is obtained. Then

$$\det \boldsymbol{A} = (-1)^r \frac{\text{product of the diagonal entries of } \boldsymbol{B}}{\text{product of scaling factors used}}, \qquad (4)$$

  where r is the number of row swaps performed.

- The recipe (4) is an efficient way of computing the determinant of a large matrix. The computational complexity of row reduction is $O\left(n^3\right)$, where n is the dimension of the square matrix.

Example

$$\begin{bmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{bmatrix} \xrightarrow{R_1 \longleftarrow \frac{1}{2} R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{bmatrix} \xrightarrow{R_3 \longleftarrow R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_3 + 7R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{bmatrix}$$

We have made two row swaps and scaled once by a factor of $\frac{1}{2}$. So, the recipe (4) allows us to compute the determinant:

$$\det \begin{bmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{bmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot (-74)}{\frac{1}{2}} = -148.$$

Takeaways

- Determinant is a function that maps the space of real-valued square matrices into the set of real numbers.
- Doing a basic row operation on $\boldsymbol{A}$ does not change $\det(\boldsymbol{A})$.
- Scaling a row of $\boldsymbol{A}$ by a scalar c multiplies the determinant by c.
- Swapping two rows of a matrix multiplies the determinant by -1.
- The determinant of the identity matrix $\boldsymbol{I}_n$ is equal to 1.
- Determinant is invariant under transpose and a basic matrix row operation.
- Co-factor approach provides a way to prove Cramer's formula, which is a system of linear equations by the ratios of determinants.
- The recipe approach to computing the determinant is more efficient than the co-factor approach.

Keywords