Portfolio Optimization

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Broad Lesson Plan

Markowitz's Important Ideas

- **Portfolio Selection**
  The investor should consider expected return a desirable thing and variance of return an undesirable thing.

- An **efficient portfolio** is the portfolio that maximizes the expected return for a given level of risk (variance of portfolio return).

- There is not only one optimal portfolio, but a set of optimal portfolios which is called the **efficient frontier**.

Great Ideas

- **Liquidity Reference** Tobin (1958)
  The efficient frontier becomes a straight line in the presence of a risk-free asset.

- Optimal portfolios correspond to a combination of the risk-free asset and one particular efficient portfolio named the **tangency portfolio**.

- Sharpe (1964)
  The process of investment choices can be broken down into two steps:
  1. the choice of a unique optimum combination of risky assets
  2. a separate choice concerning the allocation of funds between such a combination and a single risk-less asset

CAPM

- One difficulty when computing the tangency portfolio is to precisely define the vector of expected returns of the risky assets and the corresponding covariance matrix of asset returns.
- Solution: CAPM theory
  1. A portfolio's excess returns is proportional to the risk premium.
  2. If the market is in equilibrium, the prices of assets are such that the tangency portfolio *is* the market portfolio.
  3. Hence, you do not need assumptions about the expected returns, volatilities, and correlations of assets to construct the tangency portfolio.
- This major contribution of Sharpe led to the emergence of index funds and to the increasing development of **passive management**.

Setup

- Vector of weights $\boldsymbol{x}' := (x_1 \quad x_2 \quad \cdots \quad x_n)$

$$\sum_{i=1}^{n} x_i = \mathbf{1}'\boldsymbol{x} = 1$$

- Vector of asset returns $\boldsymbol{R}' := (R_1 \quad R_2 \quad \cdots \quad R_n)$

$$\boldsymbol{R}(\boldsymbol{x}) = \sum_{1=1}^{n} x_i R_i = \boldsymbol{x}'\boldsymbol{R}$$

- Expected return of assets: $\boldsymbol{\mu} := \mathbb{E}(\boldsymbol{R})$
- Covariance matrix of asset returns: $\boldsymbol{\Sigma} := \mathbb{E}\left[(\boldsymbol{R}-\boldsymbol{\mu})(\boldsymbol{R}-\boldsymbol{\mu})'\right]$

Mean and Covariance

- The expected return of the portfolio is

$$\mu(\boldsymbol{x}) := \mathbb{E}\left[\boldsymbol{R}(\boldsymbol{x})\right] = \mathbb{E}\left[\boldsymbol{x}'\boldsymbol{R}\right] = \boldsymbol{x}'\,\mathbb{E}\left[\boldsymbol{R}\right] = \boldsymbol{x}'\boldsymbol{\mu}$$

- Covariance

$$\begin{aligned}\sigma^2(\boldsymbol{x}) :=& \mathbb{E}\left[(\boldsymbol{R}(\boldsymbol{x}) - \boldsymbol{\mu}(\boldsymbol{x}))(\boldsymbol{R}(\boldsymbol{x}) - \boldsymbol{\mu}(\boldsymbol{x}))'\right] \\ =& \mathbb{E}\left[(\boldsymbol{x}'\boldsymbol{R} - \boldsymbol{x}'\boldsymbol{\mu})(\boldsymbol{x}'\boldsymbol{R} - \boldsymbol{x}'\boldsymbol{\mu})'\right] \\ =& \mathbb{E}\left[\boldsymbol{x}'(\boldsymbol{R} - \boldsymbol{\mu})(\boldsymbol{R} - \boldsymbol{\mu})'\boldsymbol{x}\right] \\ =& \boldsymbol{x}'\,\mathbb{E}\left[(\boldsymbol{R} - \boldsymbol{\mu})(\boldsymbol{R} - \boldsymbol{\mu})'\right]\boldsymbol{x} \\ =& \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}\end{aligned}$$

σ- and μ-Problems

- σ-problem
  Maximizing the expected return of the portfolio under a volatility constraint

$$\max \mu(\boldsymbol{x}) \qquad \text{subject to} \qquad \sigma(\boldsymbol{x}) \leqq \sigma^\star$$

- μ-problem
  Minimizing the volatility of the portfolio under a return constraint

$$\min \sigma(\boldsymbol{x}) \qquad \text{subject to} \qquad \mu(\boldsymbol{x}) \geqq \mu^\star$$

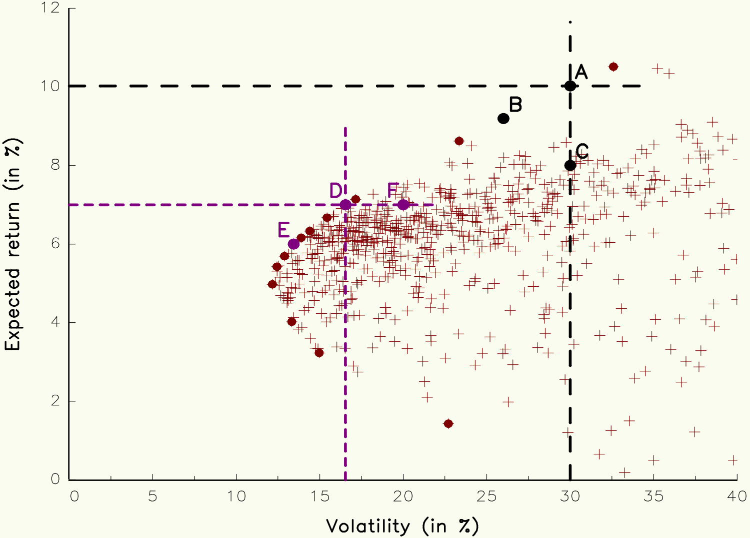
Example 1

- Four assets' expected returns are 5%, 6%, 8%, and 6% .
- Return volatilities are 15%, 20%, 25%, and 30%.
- The correlation matrix of asset returns is given by the following matrix

$$\begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}.$$

- Simulate 1,000 portfolios and report their expected return and their volatility.

Optimized Portfolios $\sigma^\star$ = 30%, $\mu^\star$ = 7%



Suppose $\sigma^\star = 30\%$

- Portfolio C could not be the solution even if it reached the volatility constraint.
- Why? Because Portfolio C is dominated by Portfolio B.
- But Portfolio B is not optimal.
- Portfolio A is optimal and is thus the solution: the expected return of the optimized portfolio is 10:02%.

Suppose $\mu^\star$ = 7%

- The efficient frontier is defined as the convex hull of the points $(\sigma(\boldsymbol{x}), \mu(\boldsymbol{x}))$ of all the possible portfolios.
- Portfolio F could not be the solution even if it reached the expected return constraint.
- Why? Because Portfolio F is dominated by Portfolio E.
- But Portfolio E cannot be the optimal solution because it is below the constraint of 7% expected return.
- The solution is Portfolio D: $\sigma\left(\boldsymbol{x}^\star\right) = 16.54\%$.

Quadratic Optimization

- Mean variance optimization

$$\boldsymbol{x}^\star(\phi) = \operatorname{argmax}\left(\boldsymbol{x}'\boldsymbol{\mu} - \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}\right) \quad \text{subject to} \quad \mathbf{1}'\boldsymbol{x} = 1 \qquad (1)$$

- ϕ: Risk aversion parameter
- If ϕ = 0, the optimized portfolio is the one that maximizes the expected return and we have $\mu(\boldsymbol{x}^\star(0)) = \mu^+$.
- If $\phi = \infty$, the optimization problem becomes

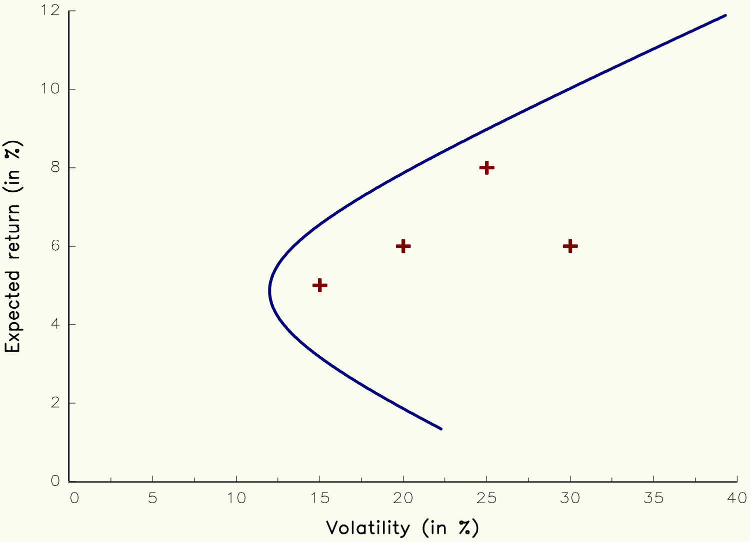
$$\boldsymbol{x}^\star(\infty) = \arg\min \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \quad \text{subject to} \quad \mathbf{1}'\boldsymbol{x} = 1,$$

and the optimized portfolio is the one that minimizes the volatility and we have $\sigma\big(\boldsymbol{x}^\star(\infty)\big) = \sigma^-$. It is called the **minimum variance (MV) portfolio**.

Solving the ϕ-problem

ϕ	$+\infty$	5.00	2.00	1.00	0.50	0.20
$x_1^\star$	72.74	68.48	62.09	51.44	30.15	−33.75
$x_2^\star$	49.46	35.35	14.17	−21.13	−91.72	−303.49
$x_3^\star$	−20.45	12.61	62.21	144.88	310.22	806.22
$x_4^\star$	−1.75	−16.44	−38.48	−75.20	−148.65	−368.99
$\mu(\boldsymbol{x}^\star)$	4.86	5.57	6.62	8.38	11.90	22.46
$\sigma(\boldsymbol{x}^\star)$	12.00	12.57	15.23	22.27	39.39	94.57

The Efficient Frontier of Markowitz



Solutions

Table: Solving the unconstrained σ-problem

$\sigma^\star$	15.00	20.00	25.00	30.00	35.00
$x_1^\star$	62.52	54.57	47.84	41.53	35.42
$x_2^\star$	15.58	−10.75	−33.07	−54.00	−74.25
$x_3^\star$	58.92	120.58	172.85	221.88	269.31
$x_4^\star$	−37.01	−64.41	−87.62	−109.40	−130.48
$\mu(\boldsymbol{x}^\star)$	6.55	7.87	8.98	10.02	11.03
ϕ	2.08	1.17	0.86	0.68	0.57

Table: Solving the unconstrained μ-problem

$\mu^\star$	5.00	6.00	7.00	8.00	9.00
$x_1^\star$	71.92	65.87	59.81	53.76	47.71
$x_2^\star$	46.73	26.67	6.62	−13.44	−33.50
$x_3^\star$	−14.04	32.93	79.91	126.88	173.86
$x_4^\star$	−4.60	−25.47	−46.34	−67.20	−88.07
$\sigma(\boldsymbol{x}^\star)$	12.02	13.44	16.54	20.58	25.10
ϕ	25.79	3.10	1.65	1.12	0.85

Adding Some Constraints

- Let Ω be the set of restrictions.

$$\boldsymbol{x}^{\star}(\phi) = \operatorname{argmax}\left(\boldsymbol{x}'\boldsymbol{\mu} - \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}\right) \quad \text{subject to} \quad \mathbf{1}'\boldsymbol{x} = 1, \boldsymbol{x} \in \Omega.$$

- The imposition of constraints will impact the set of optimized portfolios by reducing opportunity arbitrages.
- The constrained efficient frontier is located at the right of the unconstrained efficient frontier in the mean-variance map.

Short Selling

- In this case, $x_i \geqq 0$ and $\Omega = [0,1]^n$.
- The leverage measure of the portfolio $\boldsymbol{x}$

$$L(\boldsymbol{x}) := \sum_{1=1}^{n} |x_i|.$$

- With the no short-selling restriction, the leverage measure is 100% because of the “constraint’ $L(\boldsymbol{x}) = \sum_{i=1}^{n} x_i = 1$.

Without Short-Selling Constraint

Proposition

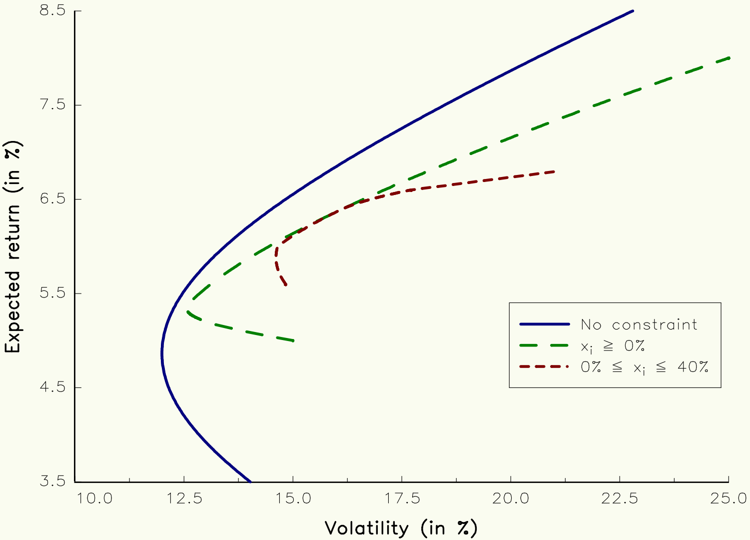
Without the no short-selling constraint, the leverage measure is larger than 100% .

- Let $x_i^- := -\min(0, x_i)$ and $x_i^+ = \max(0, x_i)$ be, respectively, the negative and positive parts of the weight x_i.
- By definition, $x_i = x_i^+ - x_i^-$, and $\sum_{i=1}^n x_i = \sum_{i=1}^n x_i^+ - \sum_{i=1}^n x_i^- = 1.$
- It follows that

$$L(\boldsymbol{x}) := \sum_{1=1}^n |x_i^+ - x_i^-| = \sum_{1=1}^n x_i^+ + \sum_{1=1}^n x_i^-$$

$$= 1 + \sum_{1=1}^n x_i^- + \sum_{1=1}^n x_i^- = 1 + 2\sum_{1=1}^n x_i^- \geqq 1.$$

Efficient Frontiers with Weight Constraints



Solving the σ-Problem with Weight Constraints

	$x_i \in \mathbb{R}$		$x_i \geq 0$		$0 \leq x_i \leq 40\%$	
$\sigma^\star$	15.00	20.00	15.00	20.00	15.00	20.00
$x_1^\star$	62.52	54.57	45.59	24.88	40.00	6.13
$x_2^\star$	15.58	-10.75	24.74	4.96	34.36	40.00
$x_3^\star$	58.92	120.58	29.67	70.15	25.64	40.00
$x_4^\star$	-37.01	-64.41	0.00	0.00	0.00	13.87
$\mu(\boldsymbol{x}^\star)$	6.55	7.87	6.14	7.15	6.11	6.74
ϕ	2.08	1.17	1.61	0.91	1.97	0.28

Lagrange Function and First-Order Condition

- The Lagrange function of the optimization problem is

$$\mathcal{L}(\boldsymbol{x};\lambda_0) = \boldsymbol{x}'\boldsymbol{\mu} - \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} + \lambda_0\big(\mathbf{1}'\boldsymbol{x} - 1\big),$$

where λ_0 is the Lagrange coefficients associated with the constraint $\mathbf{1}'\boldsymbol{x} = 1$.

- The solution $\boldsymbol{x}$ verifies the following first-order conditions:

$$\partial_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x};\lambda_0) = \boldsymbol{\mu} - \phi\boldsymbol{\Sigma}\boldsymbol{x} + \lambda_0\mathbf{1} = \mathbf{0}$$

$$\partial_{\lambda_0}\mathcal{L}(\boldsymbol{x};\lambda_0) = \mathbf{1}'\boldsymbol{x} - 1 = 0$$

Solution

- We obtain
$$\boldsymbol{x} = \phi^{-1}\boldsymbol{\Sigma}^{-1}\big(\boldsymbol{\mu} + \lambda_0\mathbf{1}\big)$$
- Because $\mathbf{1}'\boldsymbol{x} - 1 = 0$, we have
$$\mathbf{1}'\phi^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \lambda_0\big(\mathbf{1}'\phi^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{1}\big) = 1,$$
which leads to
$$\lambda_0 = \frac{1 - \mathbf{1}'\phi^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\phi^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$
- Hence, the solution $\boldsymbol{x}^*(\phi)$ is
$$\boldsymbol{x}^*(\phi) = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{1}{\phi}\cdot\frac{\big(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\big)\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \big(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\big)\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

Remarks

- What is the portfolio that corresponds to the global minimum variance?

$$\boldsymbol{x}_{\mathrm{mv}} = \boldsymbol{x}^*(\infty) = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

- For no short-selling constraint, the Lagrange function becomes

$$\mathcal{L}(\boldsymbol{x}; \lambda_0, \boldsymbol{\lambda}) = \boldsymbol{x}'\boldsymbol{\mu} - \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} + \lambda_0\big(\mathbf{1}'\boldsymbol{x} - 1\big) + \boldsymbol{\lambda}'\boldsymbol{x},$$

where $\boldsymbol{\lambda} := (\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n)$ is the vector of Lagrange coefficients associated with the constraints $x_i \geqq 0$.

When There is a Risk-Free Asset

- We have $n+1$ assets in the universe where the first n assets correspond to the risky assets and the last asset is the risk-free asset.
- Let r be the return of the risk-free asset. The return of the portfolio is

$$R(\boldsymbol{y}) = (1-\alpha)r + \alpha R(\boldsymbol{x}),$$

  where $\boldsymbol{y} = \begin{pmatrix} \alpha\boldsymbol{x} \\ 1-\alpha \end{pmatrix}$. is a vector of dimension $(n+1)$ and $\alpha \geqq 0$ is the proportion of the wealth invested in the risky portfolio.
- It follows that

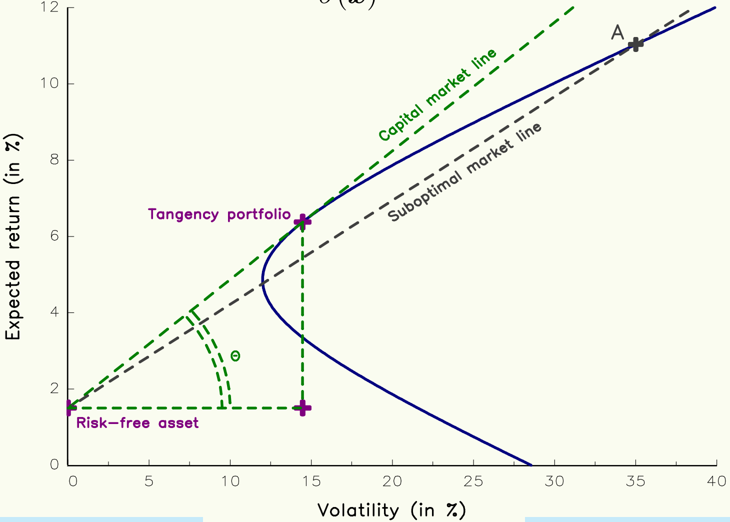
$$\mu(\boldsymbol{y}) = (1-\alpha)r + \alpha\mu(\boldsymbol{x}) = r + \alpha(\mu(\boldsymbol{x}) - r)$$

  and

$$\sigma^2(\boldsymbol{y}) = \alpha^2\sigma^2(\boldsymbol{x})$$

The Capital Market Line

- It follows that $\mu(\boldsymbol{y}) = r + \dfrac{\mu(\boldsymbol{x}) - r}{\sigma(\boldsymbol{x})}\sigma(\boldsymbol{y})$.



The Sharpe Ratio

- Let SR(x|r) be the Sharpe ratio of portfolio x, i.e.

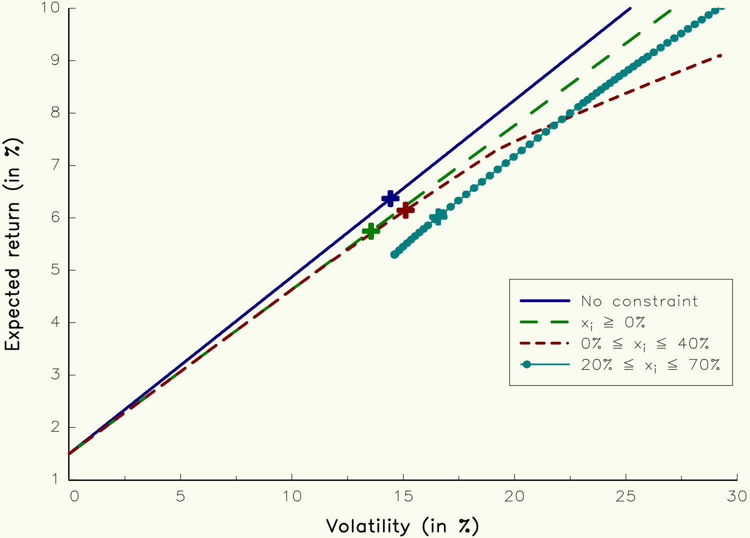
$$\mathsf{SR}(\boldsymbol{x}|r) = \frac{\mu(\boldsymbol{x}) - r}{\sigma(\boldsymbol{x})}$$

- We note that we can write the previous equation as follows:

$$\frac{\mu(\boldsymbol{y}) - r}{\sigma(\boldsymbol{y})} = \frac{\mu(\boldsymbol{x}) - r}{\sigma(\boldsymbol{x})} \iff \mathsf{SR}(\boldsymbol{y}|r) = \mathsf{SR}(\boldsymbol{x}|r)$$

- The tangency portfolio is the one that maximizes the angle θ or equivalently $\tan\theta$, which is equal to the Sharpe ratio.
- The tangency portfolio is also the risky portfolio corresponding to the maximum Sharpe ratio.
- **Any portfolio that belongs to the capital market line has the same Sharpe ratio.**

The Efficient Frontier with a Risk-Free Asset



Observations

- The efficient frontier with the risk-free asset is exactly the capital market line.
- The efficient frontier with positive weights is still a straight line. But, if $0 \leqq x_i \leqq 0.4$, the corresponding tangency portfolio is $\boldsymbol{x}^\star = (40.0\% \quad 34.7\% \quad 25.3\% \quad 0\%)$. The weight of the first risky asset has reached the upper bound equal to 40%. Hence, it is only a straight line from the risk-free asset to the tangency portfolio.
- When $0.2 \leqq x_i \leqq 0.7$, the solution is $\boldsymbol{x}^\star = (36.1\% \quad 23.9\% \quad 20.0\% \quad 20.0\%)$. The efficient frontier corresponds to a straight line in a small region.

Setup

- Consider a benchmark which is represented by a portfolio $\boldsymbol{b}$ and define
$$e := R(\boldsymbol{x}) - R(\boldsymbol{b}) = \boldsymbol{x}'\boldsymbol{R} - \boldsymbol{b}'\boldsymbol{R} = (\boldsymbol{x} - \boldsymbol{b})'\boldsymbol{R}.$$
- The expected excess return is
$$\mu(\boldsymbol{x}|\boldsymbol{b}) = \mathbb{E}[e] = (\boldsymbol{x} - \boldsymbol{b})'\boldsymbol{\mu}.$$
- The tracking error is
$$\sigma(\boldsymbol{x}|\boldsymbol{b}) = \sigma(e) = \sqrt{(\boldsymbol{x} - \boldsymbol{b})'(\boldsymbol{x} - \boldsymbol{b})}$$
- The objective
$$\boldsymbol{x}^* = \mathrm{argmax}(\boldsymbol{x} - \boldsymbol{b})'\boldsymbol{\mu} \qquad \text{such that } \mathbf{1}'\boldsymbol{x} = 1 \text{ and } \sigma(e) \leqq \sigma^*.$$

Optimization

- We transform this σ-problem into a ϕ-problem:

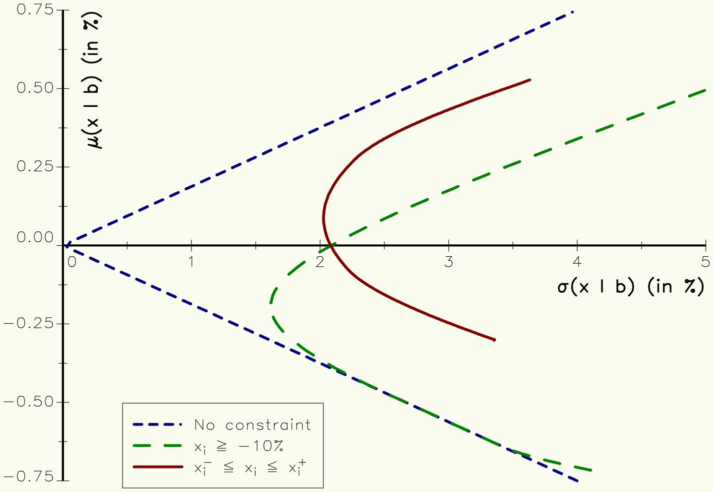
$$\boldsymbol{x}^*(\phi) = \arg\max f(\boldsymbol{x}|\boldsymbol{b}),$$

where

$$\begin{aligned} f(\boldsymbol{x}|\boldsymbol{b}) &= (\boldsymbol{x}-\boldsymbol{b})'\boldsymbol{\mu} - \frac{\phi}{2}(\boldsymbol{x}-\boldsymbol{b})'\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{b}) \\ &= \boldsymbol{x}'\big(\boldsymbol{\mu}+\phi\boldsymbol{\Sigma}\boldsymbol{b}\big) - \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{b} - \left(\frac{\phi}{2}\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{b}+\boldsymbol{b}'\boldsymbol{\mu}\right) \end{aligned}$$

- Note that the last term is a constant; it does not depend on the portfolio $\boldsymbol{x}$.
- The quadratic term can be solved using numerical algorithms.

The Efficient Frontier with a Benchmark



Illustration

- Consider Example 1 with the benchmark $\boldsymbol{b}$ = (60% 40% 20% -20%)
- The solution is a straight line when there is no restriction.
- If we impose that $x_i \geqq -10\%$, the efficient frontier is moved to the right.
- The third case assumes that the weights are between a lower bound and an upper bound: $x_i^- \leqq x_i \leqq x_i^+$ with x_i^+ = 50%. For the first three assets, the lower bound x_i^- is set to 0, whereas it is equal to -20% for the fourth asset.

Combination

- A combination of the benchmark $\boldsymbol{b}$ and the active portfolio $\boldsymbol{x}$

$$\boldsymbol{y} = (1-\alpha)\boldsymbol{b} + \alpha\boldsymbol{x},$$

with $\alpha \geqq 0$ being the proportion of wealth invested in the portfolio $\boldsymbol{x}$.

- It follows that

$$\mu(\boldsymbol{y}|\boldsymbol{b}) = (\boldsymbol{y}-\boldsymbol{b})'\boldsymbol{\mu} = \alpha\mu(\boldsymbol{x}|\boldsymbol{b})$$

$$\sigma^2(\boldsymbol{y}|\boldsymbol{b}) = (\boldsymbol{y}-\boldsymbol{b})'\boldsymbol{\Sigma}(\boldsymbol{y}-\boldsymbol{b}) = \alpha^2\sigma^2(\boldsymbol{x}|\boldsymbol{b})$$

Information Ratio

- CML is the efficient set consisting of a straight line; the maximal Sharpe ratio is not a usable criterion for portfolio allocation.
- The information ratio is defined as

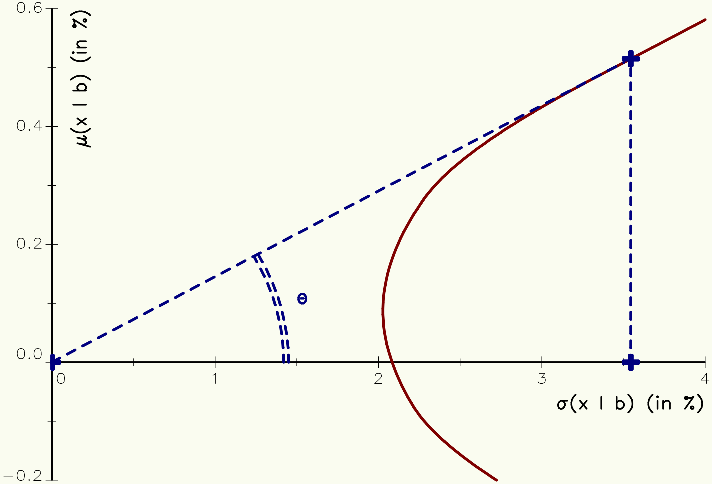
$$\mathrm{IR}(\boldsymbol{x}|\boldsymbol{b}) := \frac{\mu(\boldsymbol{x}|\boldsymbol{b})}{\sigma(\boldsymbol{x}|\boldsymbol{b})} = \frac{(\boldsymbol{x}-\boldsymbol{b})'\boldsymbol{\mu}}{\sqrt{(\boldsymbol{x}-\boldsymbol{b})'\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{b})}}$$

- We deduce that

$$\mu(\boldsymbol{y}|\boldsymbol{b}) = \mathrm{IR}(\boldsymbol{x}|\boldsymbol{b})\sigma(\boldsymbol{y}|\boldsymbol{b})$$

- It is the equation of a linear function, implying that the efficient frontier is a straight line.

Tangency Portfolio with respect to a Benchmark



Preview

- The Black-Litterman model consists in incorporating portfolio manager's views in a strategic asset allocation.
- If we omit the constraint $\mathbf{1}'\boldsymbol{x} = 1$ in (1), the solution is
$$\boldsymbol{x} = \frac{1}{\phi}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$
- In the Markowitz model, the unknown variable is the vector of weights $\boldsymbol{x}$.
- Black and Litterman (1992) assume that the initial allocation $\boldsymbol{x}_0$ corresponds to an optimal solution, and
$$\widetilde{\boldsymbol{\mu}} = \phi\boldsymbol{\Sigma}\boldsymbol{x}_0. \tag{2}$$
- Interpret $\boldsymbol{\mu}$ as the vector of expected returns, which is coherent with the portfolio $\boldsymbol{x}_0$

Result

Proposition

$$\widetilde{\boldsymbol{\mu}} = r\mathbf{1} + \text{SR}(\boldsymbol{x}_0|r)\frac{\boldsymbol{\Sigma}\boldsymbol{x}_0}{\sqrt{\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0}} \tag{3}$$

- Multiplying both sides of (2) by $\boldsymbol{x}_0'$, we obtain
$$\boldsymbol{x}_0'\widetilde{\boldsymbol{\mu}} = \phi\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0.$$
- Hence
$$\phi = \frac{\boldsymbol{x}_0'\widetilde{\boldsymbol{\mu}}}{\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0} = \frac{\text{SR}(x_0|r)}{\sqrt{\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0}} \sim \frac{\text{reward}}{\text{risk}}. \tag{4}$$
- Then
$$\begin{aligned}\widetilde{\boldsymbol{\mu}} &= \frac{\boldsymbol{x}_0'\widetilde{\boldsymbol{\mu}}}{\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0}\boldsymbol{\Sigma}\boldsymbol{x}_0 = \frac{\left(\boldsymbol{x}_0'\widetilde{\boldsymbol{\mu}} - \boldsymbol{x}_0'r\right) + \boldsymbol{x}_0'r}{\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0}\boldsymbol{\Sigma}\boldsymbol{x}_0 \\ &= \text{SR}(\boldsymbol{x}_0|r)\frac{\boldsymbol{\Sigma}\boldsymbol{x}_0}{\sqrt{\boldsymbol{x}_0'\boldsymbol{\Sigma}\boldsymbol{x}_0}} + r\mathbf{1}\end{aligned}$$

Setup of an Optimization Problem

- Black and Litterman (1992) state that $\boldsymbol{\mu}$ cannot be known with certainty. In particular, they assume that $\boldsymbol{\mu}$ is a Gaussian vector with expected returns $\widetilde{\boldsymbol{\mu}}$ and covariance matrix $\boldsymbol{\Gamma}$

$$\boldsymbol{\mu} \sim N\left(\widetilde{\boldsymbol{\mu}}, \boldsymbol{\Gamma}\right)$$

- To specify the portfolio manager's views, assume that they are given by

$$\boldsymbol{P}\boldsymbol{\mu} = \boldsymbol{Q} + \boldsymbol{\epsilon},$$

where $\boldsymbol{P}$ is a $(k \times n)$ matrix, $\boldsymbol{Q}$ is a $(k \times 1)$ vector, and $\boldsymbol{\epsilon} \sim N(0, \boldsymbol{\Omega})$ is a Gaussian vector of dimension k.

Optimization Problem

- The Markowitz optimization problem becomes, under the condition of $\mathbf{1}'\boldsymbol{x} = 1$,

$$\boldsymbol{x}^*(\phi) = \arg\max \boldsymbol{x}'\overline{\boldsymbol{\mu}} - \frac{\phi}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}. \tag{5}$$

  Here, $\overline{\boldsymbol{\mu}}$ is the vector of expected returns conditional on the views:

$$\overline{\boldsymbol{\mu}} := \mathbb{E}\left[\boldsymbol{\mu}\middle|\text{views}\right] = \mathbb{E}[\boldsymbol{\mu}|\boldsymbol{P}\boldsymbol{\mu} = \boldsymbol{Q} + \boldsymbol{\epsilon}].$$

- To compute $\overline{\boldsymbol{\mu}}$, we consider the random vector $(\boldsymbol{\mu}; \boldsymbol{\nu} = \boldsymbol{P}\boldsymbol{\mu} - \boldsymbol{\epsilon})$

$$\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} \sim N\left(\begin{pmatrix} \widetilde{\boldsymbol{\mu}} \\ \boldsymbol{P}\widetilde{\boldsymbol{\mu}} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Gamma}\boldsymbol{P}' \\ \boldsymbol{P}\boldsymbol{\Gamma} & \boldsymbol{P}\boldsymbol{\Gamma}\boldsymbol{P}' + \boldsymbol{\Omega} \end{pmatrix}\right)$$

Conditional Expectation

- We apply the conditional expectation formula

$$\overline{\boldsymbol{\mu}} = \mathbb{E}\left[\boldsymbol{\mu}\middle|\boldsymbol{\nu} = \boldsymbol{Q}\right] = \widetilde{\boldsymbol{\mu}} + \boldsymbol{\Gamma}\boldsymbol{P}'\left(\boldsymbol{P}\boldsymbol{\Gamma}\boldsymbol{P}' + \boldsymbol{\Omega}\right)^{-1}\left(\boldsymbol{Q} - \boldsymbol{P}\widetilde{\boldsymbol{\mu}}\right) \tag{6}$$

- The conditional expectation $\overline{\boldsymbol{\mu}}$ has two components:
  1. The first component corresponds to the vector of **implied expected returns** $\widetilde{\boldsymbol{\mu}}$.
  2. The second component is a correction term which takes into account the **disequilibrium** $(\boldsymbol{Q} - \boldsymbol{P}\widetilde{\boldsymbol{\mu}})$ between the manager views and the market views.

Algorithm of the Black-Litterman Model

1. Estimate the empirical covariance matrix $\widehat{\boldsymbol{\Sigma}}$.
2. Given the current portfolio, deduce the vector $\widetilde{\boldsymbol{\mu}}$ of implied expected returns, with (3).
3. Specify the views and the associated level of confidence, i.e., define the $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{\Omega}$ matrices.
4. Given Γ, compute the conditional expectation $\overline{\boldsymbol{\mu}}$ using (6).
5. Finally, perform the portfolio optimization (5) with $\overline{\boldsymbol{\mu}}$, and ϕ given by (4).

Difficulty and Solution

- The main difficulty lies in specifying the covariance matrix $\boldsymbol{\Gamma}$.
- A solution is to define $\boldsymbol{\Gamma} = \tau\boldsymbol{\Sigma}$ and to calibrate τ in order to target a tracking error volatility.
- As an example, suppose that the portfolio manager has an absolute view on the first asset and a relative view on the second and third assets.
- We specify the matrices $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{\Omega}$ as follows:

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} q_1 \\ q_{2-3} \end{pmatrix}, \text{ and } \boldsymbol{\Omega} = \begin{pmatrix} \varpi_1^2 & 0 \\ 0 & \varpi_{2-3}^2 \end{pmatrix}$$

Portfolio # 1

- Suppose q_1 = 4% and q_{2-3} = -1%.
- It means that the portfolio manager believes that the expected return of the first asset is 4% and that the third asset will outperform the second asset by 1% on average.
- Suppose also that $\overline{\omega}_1$ = 10% and $\overline{\omega}_{2-3}$ = 5%, meaning that the level of confidence is higher for the relative view than for the absolute view.
- Suppose $\tau = 1$.

Other Portfolios with respect to Portfolio 1

- Portfolio # 2: q_1 = 7% instead
- Portfolio # 3: $\overline{\omega}_1 = \overline{\omega}_{2-3}$ = 20% instead
- Portfolio # 4: τ = 10% instead
- Portfolio # 5: τ = 1% instead

	#0	#1	#2	#3	#4	#5
$x_1^\star$	40.00	33.41	51.16	36.41	38.25	39.77
$x_2^\star$	30.00	51.56	39.91	42.97	42.72	32.60
$x_3^\star$	20.00	5.46	0.00	10.85	9.14	17.65
$x_4^\star$	10.00	9.58	8.93	9.77	9.89	9.98
$\sigma\left(\boldsymbol{x}^\star \mid \boldsymbol{x}_0\right)$	0.00	3.65	3.67	2.19	2.18	0.45

Calibration of Parameter τ

- Target a tracking error volatility $\sigma^\star$.
- For example, if $\sigma^\star$ = 2%, the optimized portfolio is between portfolios #4 ($\sigma\left(\boldsymbol{x}^\star \mid \boldsymbol{x}_0\right)$ = 2.18%) and # 5 ($\sigma\left(\boldsymbol{x}^\star \mid \boldsymbol{x}_0\right)$ = 0.45%), meaning that the optimal value of τ is between 10% and 1%.
- Using a bisection algorithm, we obtain τ = 5.2% and the optimal portfolio is

$$\boldsymbol{x}^\star = (36.80\% \;\; 41.83\% \;\; 11.58\% \;\; 9.79\%).$$

Takeaways

- Quadratic optimization problem with a risk-aversion parameter
- σ-problem versus μ-problem
- Unconstrained and constrained optimizations lead to different efficient frontiers.
- Leverage ratio
- Capital Market Line appears when there is a risk-free interest rate.
- Optimization with respect to a benchmark
- The Black-Litterman model is quite popular.

Reference

- Introduction to Risk Parity and Budgeting by Thierry Roncalli (2013)