Volatility Indexes

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Lesson Plan

Two Uncertainties

Market Uncertainty

- $\rightarrow$ Price fluctuation $\Longrightarrow$ volatility
- $\rightarrow$ Correlation of volatility with return

Volatility Uncertainty

- $\rightarrow$ Volatility is also fluctuating
- $\rightarrow$ Volatility clusters
- $\rightarrow$ *Ex ante* increase or decrease in volatility is a risk

Important Questions

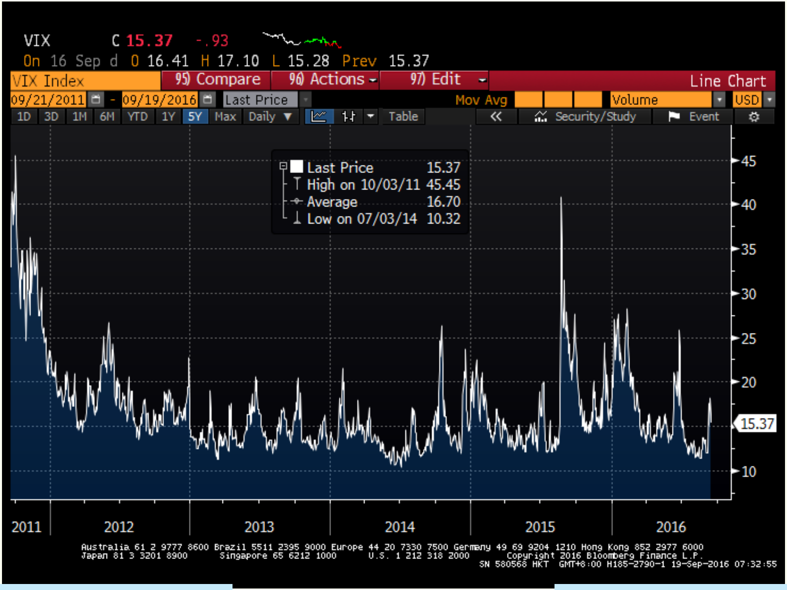
→ How to forecast *ex ante* volatility?

- Volatility index — “fear gauge”
- VIX, VXD, VXN, RVX, VDAX-NEW, VSMI, VSTOXX, ...

→ How to estimate the **volatility risk premium**?

- Investors who bear the volatility risk demand a volatility risk premium.
- Investors who don’t want to bear the higher volatility forecasted pay the volatility risk premium.

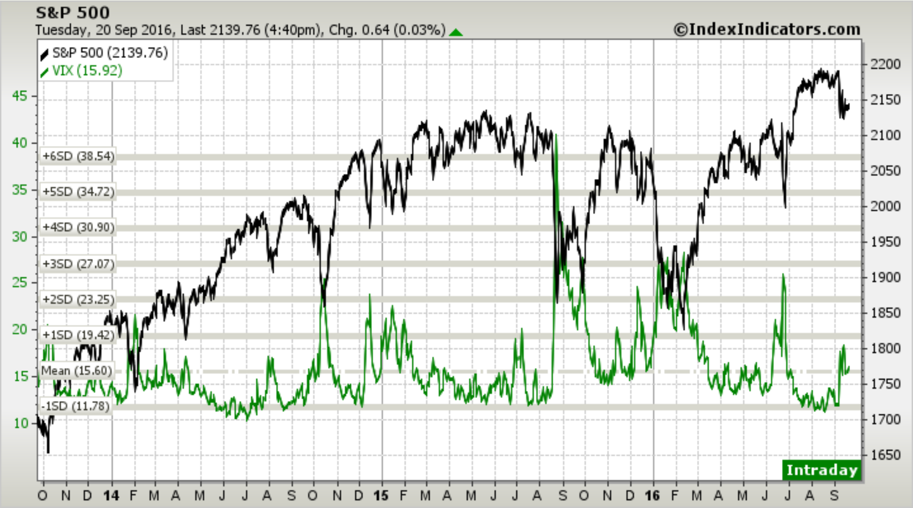
VIX: Fear Gauge?



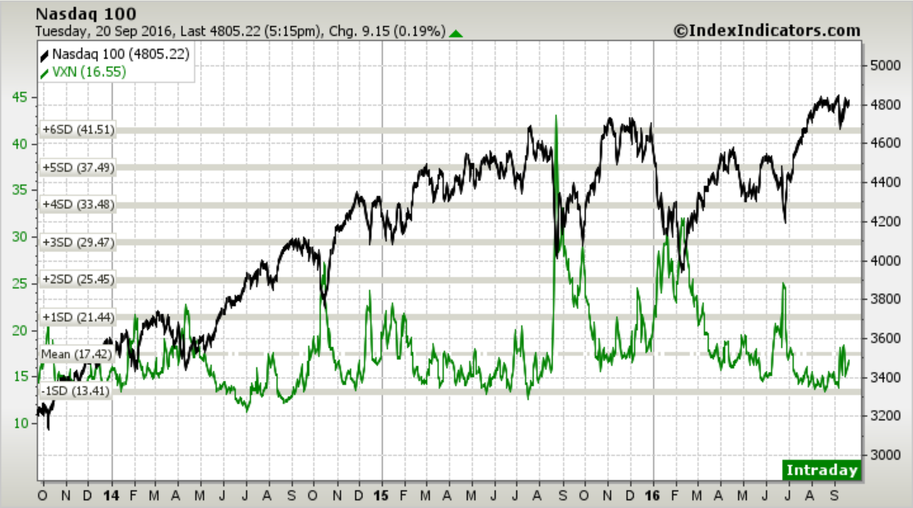
What is VIX?

- Created in 1993, VIX is the ticker symbol for the CBOE Volatility Index for S&P 500 Index.
- VIX quantifies option traders' expectation of *future* volatility for the next 30 calendar days.
- The old version of VIX relied on the Black-Scholes model to back out an **implied volatility** for each of the 8 options that are near-the-money. Old VIX is the average of these implied volatilities.
- Current new version is **model-free**, and uses as many out-of-the-money S&P 500 index options as possible.
- Why called the "fear gauge"?
  - Contributions of OTM put options are larger than OTM call options.
  - S&P 500 Index tends to be lower when VIX is higher.

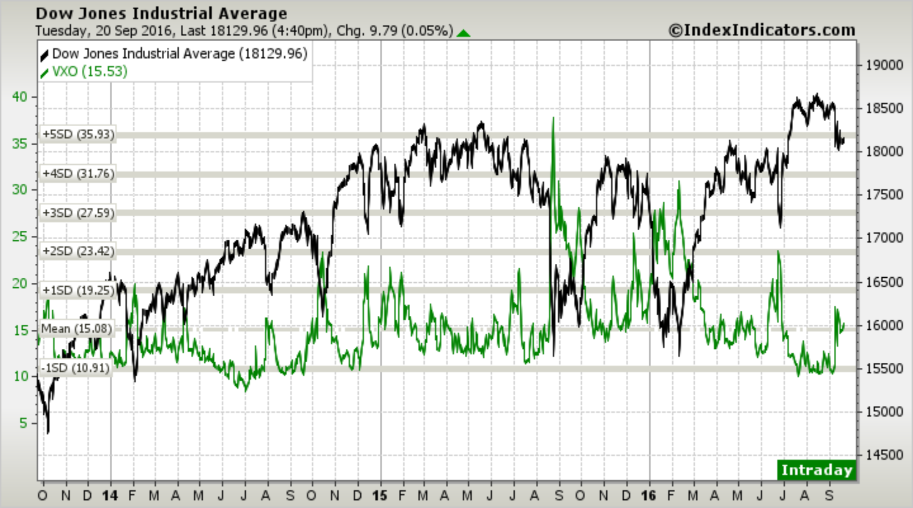
Relation with the Underlying S&P 500 Index



VXN and Nasdaq 100 Index



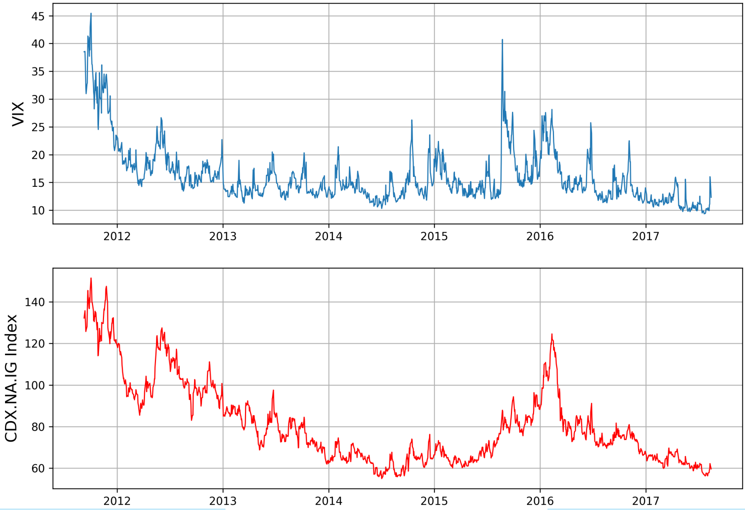
VXO and Dow Jones Industrial Average Index



VIX in Investment & Trading

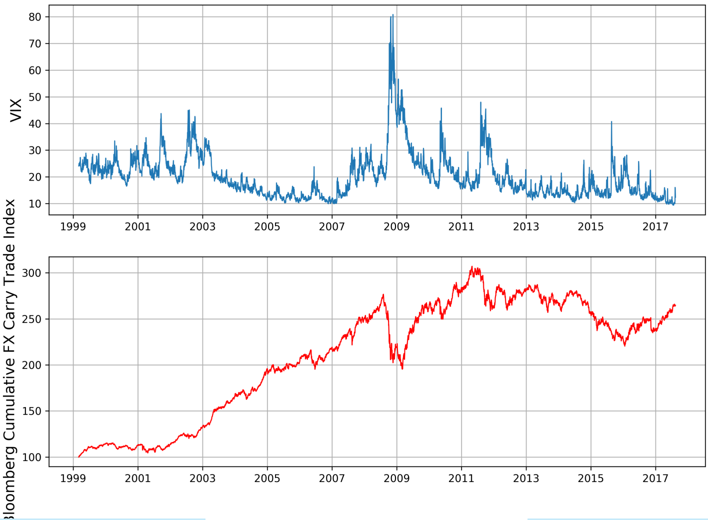
- The popularity of VIX allows CBOE to roll out futures and options on VIX.
- Speculation on the future level of volatility in a pure manner.
  - short the VIX futures when VIX is “unusually” high
  - long the VIX futures when VIX is “unusually” low
- Hedge against long equity exposure.
- Hedge against a high correlation market condition, which typically makes stock selection more difficult.
- Tracking of **aggregate credit spread**
- Tracking of **carry trade** benchmark (read **Carry Trade Defined**)

VIX and CDX



VIX
45
40
35
30
25
20
15
10
2012
2013
2014
2015
2016
2017
CDX.NA.IG Index
140
120
100
80
60
2012
2013
2014
2015
2016
2017

VIX and Carry Trade Benchmark



Applications of VIX

- Volatility becomes a tradable “asset class”.
  - CBOE offers futures and options on VIX—revenue generation for the exchange.
  - Speculation: Express a view on future volatility through trading.
  - Hedging: Reduction of NAV fluctuation.
- VIX^2 as the fair value for a 30-day **variance swap**. The payoff function (same as P&L in this case) of this forward contract of amount A for the buyer is, at maturity $T = 30$,

$$\text{P\&L of Buyer}_T = A \times \left(\textbf{Realized Variance} - \text{VIX}^2\right),$$

where the realized variance is the variance of future daily returns from day 0 up to T.

Implied Volatility

- Implied volatility used to be model-dependent.
  - Black-Scholes option pricing formula
  - Binomial tree
- Model risk
  - All models are wrong....

    — George Box
  - A **smile surface** that extends well into the **wings**, which are suspect of model risk
- Model-free approach to implied volatility
  - VIX — discrete computations
  - Academics — "smooth" computations

Model-Free Variance σ^2_{MF}

- Direct computation given the midquotes of puts and call options

$$\sigma^2_{\text{MF}} T := \mathbb{E}_0^{\mathbb{Q}}\big(V(0,T)\big)$$

$$= 2e^{r_0 T}\left(\int_0^{F_0} \frac{p(X,S_0,T)}{X^2}dX + \int_{F_0}^{\infty} \frac{c(X,S_0,T)}{X^2}dX\right) \quad (1)$$

- Strike price: X
- Time to maturity: T
- Underlying asset price at time 0: S_0
- Forward price of the underlying asset: F_0
- Risk-free rate of tenor T: r_0
- European put's midquote: $p(X, S_0, T)$
- European call's midquote: $c(X, S_0, T)$

Features of Model-Free Approach

- No requirement for an option pricing model

  $\Longrightarrow$ No model risk!

- No worry about parameters
  - The only exogenous inputs are risk-free interest rate and dividend yields

- No bias
  - σ_{MF} reflects volatility across all out-of-the-money strike prices and thus reflects the option skew

- Uses both put and call options

  $\Longrightarrow$ σ_{MF} is less sensitive to individual option prices.

First Principle for Option Prices

- Recall that the first principle involves a risk-free rate r_0.
- The price P_0 today and the expected payoff Z_T, which will be settled T years from today, are related by the first principle:

$$P_0 = e^{-r_0 T}\,\mathbb{E}_0^{\mathbb{Q}}(Z_T).$$

- For European call option, the payoff is

$$(S_T - X)^+ =: \max(S_T - X, 0).$$

So in general, given the strike price X,

$$c_0 = e^{-r_0 T}\,\mathbb{E}_0^{\mathbb{Q}}\left((S_T - X)^+\right).$$

- For European put option,

$$p_0 = e^{-r_0 T}\,\mathbb{E}_0^{\mathbb{Q}}\left((X - S_T)^+\right).$$

Variance as a Difference of Two Returns

- With R_t being the simple return, Pre-U Maclaurin's series gives

$$\ln\left(1+R_t\right) = R_t - \frac{1}{2}R_t^2 + O(R_t^3).$$

- In other words, the following approximation holds because at daily frequency or higher, R_t is generally very small.

$$R_t^2 \approx 2\Big(R_t - \ln\left(1+R_t\right)\Big). \qquad (2)$$

- Since the mean $\mathbb{E}\left(R_t\right) \approx 0$, $\mathbb{E}\left(R_t^2\right) \approx \mathbb{V}\left(R_t\right)$, i.e. R_t^2 may be regarded as the variance σ_t^2 of time t.
- Insight: Twice the difference between the simple return R_t and the log return $\ln\left(1+R_t\right)$ is the variance.
- Is it guaranteed that $R_t - \ln\left(1+R_t\right) \geq 0$? YES!

Integrated Variance

- Next, we consider the integrated variance $V(0,T)$ defined as

$$V(0,T) := \int_0^T \sigma_t^2 \, dt.$$

- The variance $V(0,T)$ is the sum of instantaneous variances σ_t^2 realized over time 0 to time T.

- From (2)

$$\int_0^T \sigma_t^2 \, dt = 2\int_0^T R_t \, dt - 2\int_0^T \ln(1+R_t) \, dt. \quad (3)$$

Integrated Variance as Model-Free Variance

- Now, under the risk neutral measure $\mathbb{Q}$, and assuming a risk-free rate r_0 that remains constant from time 0 to time T,

$$\mathbb{E}_0^{\mathbb{Q}}\left(\int_0^T R_t\,dt\right) = \int_0^T \mathbb{E}_0^{\mathbb{Q}}(R_t)\,dt = \int_0^T r_0\,dt = r_0 T.$$

- On the other hand, **telescoping sum**

$$\ln\left(\frac{S_1}{S_0}\right) + \ln\left(\frac{S_2}{S_1}\right) + \cdots + \ln\left(\frac{S_{T-1}}{S_{T-2}}\right) + \ln\left(\frac{S_T}{S_{T-1}}\right) = \ln\left(\frac{S_T}{S_0}\right)$$

for tiny time interval $\Delta t = 1$ unit results in

$$\mathbb{E}_0^{\mathbb{Q}}\left(\int_0^T \ln(1+R_t)\,dt\right) = \mathbb{E}_0^{\mathbb{Q}}\left(\ln\left(\frac{S_T}{S_0}\right)\right),$$

since $1 + R_t = \dfrac{S_{t+\Delta t}}{S_T}$.

Under Risk-Neutral Measure

- Putting all terms together, we have

$$\begin{aligned}\sigma^2_{\mathsf{MF}} T &:= \mathbb{E}^{\mathbb{Q}}_0\big(V(0,T)\big) = \mathbb{E}^{\mathbb{Q}}_0\left(\int_0^T \sigma_t^2\, dt\right) \\ &= 2r_0 T - 2\mathbb{E}^{\mathbb{Q}}_0\left(\ln\left(\frac{S_T}{S_0}\right)\right). \end{aligned} \quad (4)$$

Forward Price

- For the second term on the right-hand side in (4), we consider F_0 known at time $t = 0$, and we express $\ln(S_T/F_0)$ as

$$\begin{aligned} \ln\left(\frac{S_T}{F_0}\right) &= \ln S_T - \ln F_0 - S_T\left(\frac{1}{F_0} - \frac{1}{S_T}\right) + \frac{S_T}{F_0} - 1 \\ &= \int_{F_0}^{S_T} \frac{1}{X}\,dX - S_T\int_{F_0}^{S_T}\frac{1}{X^2}\,dX + \frac{S_T}{F_0} - 1 \\ &= -\int_{F_0}^{S_T}\frac{S_T - X}{X^2}\,dX + \frac{S_T}{F_0} - 1\,. \end{aligned} \tag{5}$$

- For any $z > -1$, $\ln(1+z)$ is a strictly concave function, hence $\ln(1+z) < z$. The left side of equation (5) is $\ln(1+z)$ with $z := \dfrac{S_T}{F_0} - 1.$
- It follows that the integral $\displaystyle\int_{F_0}^{S_T}\frac{S_T - X}{X^2}\,dX$ equals $z - \ln(1+z)$ and hence is strictly positive.

Strictly Positive

- We can then rewrite the integral as

$$\begin{aligned}\int_{F_0}^{S_T} \frac{S_T - X}{X^2}\, dX &= 1_{S_T > F_0} \int_{F_0}^{S_T} \frac{S_T - X}{X^2}\, dX + 1_{S_T < F_0} \int_{F_0}^{S_T} \frac{S_T - X}{X^2}\, dX \\ &= 1_{S_T > F_0} \int_{F_0}^{S_T} \frac{S_T - X}{X^2}\, dX - 1_{S_T < F_0} \int_{S_T}^{F_0} \frac{S_T - X}{X^2}\, dX \\ &= 1_{S_T > F_0} \int_{F_0}^{S_T} \frac{S_T - X}{X^2}\, dX + 1_{S_T < F_0} \int_{S_T}^{F_0} \frac{X - S_T}{X^2}\, dX \\ &= \int_{F_0}^{\infty} \frac{(S_T - X)^+}{X^2}\, dX + \int_{0}^{F_0} \frac{(X - S_T)^+}{X^2}\, dX\,. \qquad (6)\end{aligned}$$

- In the last step, we have used the fact that the asset price S_T, which is unknown at time $t = 0$, can potentially attain a low value 0, or appreciate substantially to a high value ∞.

Risk-Neutral Expectation

- In view of (6), (5) becomes, under the risk-neutral measure $\mathbb{Q}$,

$$\mathbb{E}_0^{\mathbb{Q}}\left(\ln\left(\frac{S_T}{F_0}\right)\right) = -\,e^{r_0 T}\int_{F_0}^{\infty}\frac{c(S_0,X,T)}{X^2}\,dX - e^{r_0 T}\int_0^{F_0}\frac{p(S_0,X,T)}{X^2}\,dX + \mathbb{E}_0^{\mathbb{Q}}\left(\frac{S_T}{F_0}-1\right)$$
$$= -\,e^{r_0 T}\int_{F_0}^{\infty}\frac{c(X,S_0,T)}{X^2}\,dX - e^{r_0 T}\int_0^{F_0}\frac{p(X,S_0,T)}{X^2}\,dX. \quad (7)$$

- To arrive at this result, $\mathbb{E}_0^{\mathbb{Q}}(S_T) = F_0$ has been applied:

$$\mathbb{E}_0^{\mathbb{Q}}\left(\frac{S_T}{F_0}-1\right) = \frac{\mathbb{E}_0^{\mathbb{Q}}(S_T)}{F_0} - 1 = 0.$$

Last Step

- Finally, we write

$$\ln\frac{S_T}{S_0} = \ln\frac{S_T}{F_0} + \ln\frac{F_0}{S_0}. \tag{8}$$

- Substituting (7) into (4), we obtain

$$\sigma^2_{\mathsf{MF}}T = 2r_0T + 2e^{r_0T}\left(\int_{F_0}^{\infty}\frac{c(X,S_0,T)}{X^2}\,dX + \int_0^{F_0}\frac{p(X,S_0,T)}{X^2}\,dX\right) - 2\ln\frac{F_0}{S_0},$$

- Since $F_0 = e^{r_0T}S_0$, the first and last terms cancel out and the model-free formula (1) is obtained.

□

Advantages and Limitations

- The model-free approach incorporates information from out-of-the-money puts and calls (with respect to the forward price F_0) to produce a *single* implied volatility σ_{MF} for a given maturity.

- Given the weight $\dfrac{1}{K^2}$, out-of-the-money puts contribute more to σ_{MF}, hence "fear" gauge.

- The model-free approach to implied volatility is applicable only for European options.

- Equity index options are typically European but stock options are American.

Issues in Implementation

- Strike price is not continuous but discrete.
- Strike prices in the option chain are not from 0 to ∞.
- Most options are illiquid and most have only ask prices but not bid prices.

Market Reality Example

Calls			Puts	
Bid	Ask	Strike	Bid	Ask
25.9	30.3	67.5	0.1	4.4
23.5	27.9	70	0.05	4.4
21	25.4	72.5	0.05	4.4
18.5	22.9	75	0.05	4.4
16	20.4	77.5	0.05	4.4
13.5	17.9	80	0.05	4.4
11.1	15.5	82.5	0.05	4.4
8.6	13	85	0.1	4.4
6.1	10.5	87.5	0.05	4.4
3.9	8.3	90	0.3	4.4
1.7	6.1	92.5	0.3	4.4
0.1	4.5	95	0.9	4.4
0.8	4.4	97.5	0.4	4.8
0.05	4.4	100	2.5	6.5
0.05	4.4	102.5	4.7	9.1
0.1	4.4	105	7.1	11.5
0.1	4.4	107.5	9.6	14
0.05	4.4	110	12.1	16.5
0	4.4	112.5	14.5	18.9
0	4.4	115	17	21.4
0	4.4	117.5	19.5	23.9

- **Option chain** of BKX, KBW Nasdaq Bank Index (@ 95.57)
- Expiration: Sep 15, 2017
- Source: Marketwatch.com, as at end of Aug 15, 2017
- Discrete strike price
- Limited strike range
  - lowest strike L=\$30
  - highest strike H=\$150
- Not liquid
- But quotes are firm, ready for trades

CBOE's Implementation

- According to a CBOE's **white paper**, the generalized formula used in the VIX calculation is

$$\sigma^2_{\text{CBOE}} = \frac{2e^{r_0 T}}{T} \sum_{i=1} \frac{\Delta K_i}{K_i^2} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2,$$

where

- K_0 is the first strike below the forward index level, F;
- K_i is the strike price of the i-th out-of-the-money option; a call if $K_i > K_0$; and a put if $K_i < K_0$; both put and call if $K_i = K_0$;
- ΔK_i is the interval between strike prices–half the difference between the strike on either side of K_i;
- $Q(K_i)$ is the midpoint of the bid-ask spread for each option with strike K_i.

CBOE's Method in Detail

- SPX option selection criteria
  - Out-of-the-money with respect to K_0
  - Non-zero bid price
  - Once two puts (calls) with consecutive strike prices are found to have zero bid prices, no puts (calls) with lower (higher) strikes are considered for inclusion.

- Determine the forward SPX level, F
  - Identify the strike price K_s at which the absolute difference between the call and put prices is smallest. Then

$$F = K_s + e^{r_0 T}\big(c(K_s) - p(K_s)\big)$$

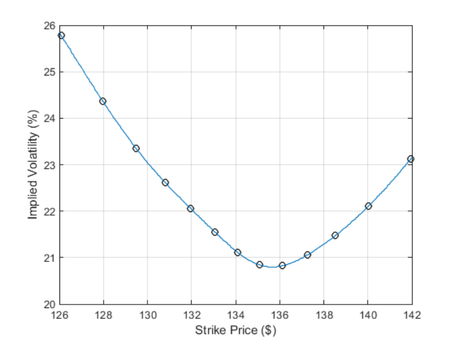
  - Determine K_0 as the strike price immediately below F

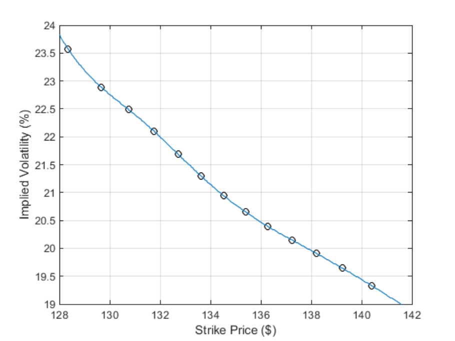
Current Academic Practice

- Compute the midquotes.
- Convert the midquotes into implied volatilities with either Black-Scholes or binomial tree pricing model.
- Interpolate the implied volatilities with a spline into a smirk and smile (read **TheOptionsGuide.com**).
- Sample as many synthetic implied volatilities as possible from the volatility smirk/smile.
- Apply the pricing model to obtain the synthetic option prices.

Interpolation of Implied Volatilities

OptionMetrics' IvyDB US database





Fully Model-Free and Exact

- Option pricing models at all stages of computation are not involved.
- Rely exclusively on **put-call parity** $c_0 - p_0 = S_0 e^{-qT} - X e^{-rT}$, where q is the dividend yield.
- The **synthetic option** $o^k(X,T)$ over any small sub-interval $(X_k, X_{k+1}]$ of strikes is represented locally as a cubic polynomial function:

$$o^k(X,T) = s_1^k X^3 + s_2^k X^2 + s_3^k X + s_4^k. \quad (9)$$

- Every cubic spline is defined by its coefficients s_1^k to s_4^k. Integration over each sub-interval $(X_k, X_{k+1}]$ admits a closed form expression:

$$\int_{X_k}^{X_{k+1}} \frac{o^k(X,T)}{X^2}\, dX = s_1^k \frac{X_{k+1}^2 - X_k^2}{2} + s_2^k (X_{k+1} - X_k) + s_3^k \ln\left(\frac{X_{k+1}}{X_k}\right) - s_4^k \left(\frac{1}{X_{k+1}} - \frac{1}{X_k}\right). \quad (10)$$

Exact Representation

- Let there be M sub-intervals for the integration from L to F_0 for puts, and N sub-intervals for the integration from F_0 to H for calls in the model-free formula, (1). We obtain an exact representation of (1) as follows:

$$
\begin{aligned}
\sigma^2_{\mathsf{MF}} T \;=\; & 2e^{rT}\left(\sum_{i=-M}^{-1} p_1^i \frac{X_{i+1}^2 - X_i^2}{2} + p_2^i (X_{i+1} - X_i)\right. \\
& \left. + p_3^i \ln\left(\frac{X_{i+1}}{X_i}\right) - p_4^i\left(\frac{1}{X_{i+1}} - \frac{1}{X_i}\right)\right) \\
& + 2e^{rT}\left(\sum_{i=0}^{N-1} c_1^i \frac{X_{i+1}^2 - X_i^2}{2} + c_2^i (X_{i+1} - X_i)\right. \\
& \left. + c_3^i \ln\left(\frac{X_{i+1}}{X_i}\right) - c_4^i\left(\frac{1}{X_{i+1}} - \frac{1}{X_i}\right)\right) \qquad (11)
\end{aligned}
$$

No Risk-Free Arbitrage

❄ For three strike prices X_a, X_b, and X_c such that $X_a < X_b < X_c$ the conditions necessary for the absence of arbitrage are

(I) Price monotonicity

$$p_a \leq p_b\,; \qquad c_b \leq c_a\,. \tag{12}$$

(II) Gradient bounds

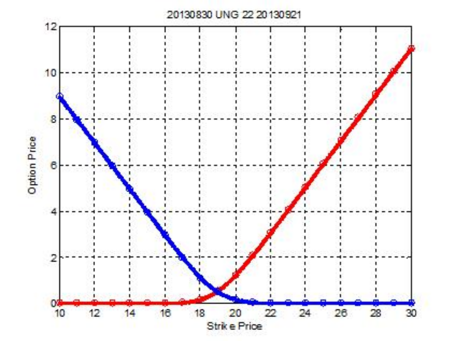
$$0 \leq \frac{p_b - p_a}{X_b - X_a} \leq 1\,; \qquad -1 \leq \frac{c_b - c_a}{X_b - X_a} \leq 0\,. \tag{13}$$

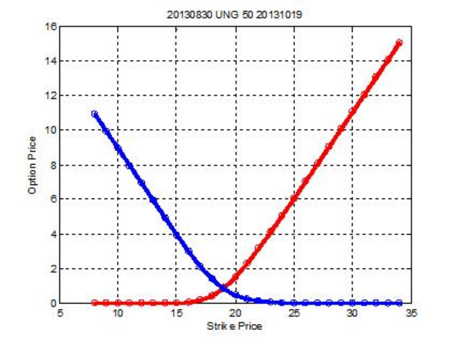
(III) Convexity

$$\frac{p_b - p_a}{X_b - X_a} \leq \frac{p_c - p_b}{X_c - X_b}\,; \qquad \frac{c_b - c_a}{X_b - X_a} \leq \frac{c_c - c_b}{X_c - X_b}\,. \tag{14}$$

Interpolation of Prices with Three Constraints

OptionMetrics' IvyDB US database





Volatility Index

- To obtain the annualized volatility index σ for a fixed time horizon or **constant maturity** T, we interpolate the model-free variances $\sigma_a^2 T_a$ and $\sigma_b^2 T_b$ with $T_a < T < T_b$.
- At time 0, following CBOE's practice, the model-free volatility index σ is obtained by linear interpolation as follows:

$$\sigma^2 T = \sigma_a^2 T_a \frac{T_b - T}{T_b - T_a} + \sigma_b^2 T_b \frac{T - T_a}{T_b - T_a}. \tag{15}$$

- The Actual/365 day-count convention is used to annualize the variance.
- Example: In Slide 36, we have $T_a = 22/365$ and $T_b = 50/365$. We obtain $\sigma_a = 20.81\%$ and $\sigma_b = 24.20\%$. For 30-day constant maturity, i.e., $T = 30/365$, applying (15) results in a model-free volatility index σ of 22.49% for August 30, 2013.

Takeaways

- VIX, known as the “fear gauge”, has become an important index in the financial market.
- Volatility, like credit, is now a tradable “asset class”.
- Implied volatilities such as VIX are forward looking, i.e., ex ante.
- Volatility risk premium can be estimated by the P&L of variance swaps.
- Existing methods are not fully model-free.
- Our fully model-free method is better and robust.
  - No risk-free arbitrary opportunity through the constraints
  - Exact

Acceptance of a $\mathbb{Q}$uantitative Finance Model

In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant.

Fischer Black (1986)

Assignment

Question A

The risk-free zero-coupon interest rate is 0.97%, and the dividend yield is assumed to be zero. Refer to Slide 28.

1. Compute the midquotes of ITM calls to obtain the corresponding OTM put option prices for strike prices from \$67.5 to \$97.5.
2. Likewise, compute the midquotes of ITM puts to obtain the corresponding OTM call option prices for strike prices from \$95 to \$117.5.
3. Select a pair of near-the-money put option prices (midquote and/or the price obtained from ITM call option). You have to exercise judgment on which of these prices should be less "incorrect". Likewise, select a pair of near-the money call options. Using linear interpolation, find the implied forward price F_0.

Assignment (cont'd)

Question B

Revise and reflect on all the materials covered so far. Write in one page (A4 size paper) about a few specific concepts that are most difficult and you are struggling with. What are the gaps that you think exist between your current level of math proficiency and finance literacy in relation to a concept difficult for you?